KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS

VOLUME XXI

N°. 5.

President: Prof. H. A. LORENTZ.
Secretary: Prof. P. ZEEMAN.

(Translated from: "Verslag van de gewone vergaderingen der Wis- en Natuurkundige Afdeeling," Vol. XXVI and XXVII).

CONTENTS.

D. J. HULSHOFF POL: "Our equilibrium-organ". (Communicated by Prof C. WINKLER), p. 626.
D. J. HULSHOFF POL: "Cerebellar ataxia as disturbance of the equilibrium-sensation". (Communicated by Prof. C. WINKLER), p. 637.
J. J. VAN LAAR: "On the Heat of Dissociation of Di-atomic Gases in Connection with the Increased Valency-Attractions $\sqrt{A}$ of the Free Atoms". (Communicated by Prof. H. A. LORENTZ), p. 644.
P. G. CATH: "On the measurement of very low temperatures. XXIX. Vapour-pressures of oxygen and nitrogen for obtaining fixed points on the temperature-scale below 0° C". (Communicated by Prof. H. KAMERLINGH ONNES), p. 656.
F. E. C. SCHEFFER: "On Phenyl Carbaminic Acid and its Homologues". (Communicated by Prof. BÖESEKEN), p. 664.
J. J. HAAK and R. SISSINGH: "Experimental Inquiry into the Nature of the Surface-Layers in the Reflection by Mercury, and into the Difference in the Optical Behaviour of Liquid and Solid Mercury". (Communicated by Prof. H. A. LORENTZ), p. 678.
F. M. JAEGER and WILLIAM THOMAS B. SC.: "Investigations on PASTEUR's Principle concerning the Relation between Molecular and Crystallonomical Dissymmetry: VIII. On the spontaneous Fission of racemic Potassium-Cobalti-Oxalate into its optically-active Antipodes", p. 693.
L. E. J. BROUWER: "Ueber eineindeutige, stetige Transformationen von Flächen in sich". (Sechste Mitteilung), p. 707.
EUG. DUBOIS: "The Significance of the Size of the Neurone and its Parts". (Communicated by Prof. H. ZWAARDEMAKER), p. 711.
A. PANNEKOEK: "Expansion of a cosmic gassphere, the new stars and the Cepheids". (Communicated by Prof. W. DE SITTER), p. 730.
J. D. VAN DER WAALS JR.: "On the Theory of the Friction of Liquids. (Communicated by Prof. J. D. VAN DER WAALS), p. 743.
A. B. DROOGLEEVER FORTUYN: "On two Nerves of Vertebrates agreeing in Structure with the Nerves of Invertebrates". (Communicated by Prof. J. BOEKE), p. 756.
N. G. W. H. BEEGER: "Ueber die Teilkörper des Kreiskörpers $K\left(\frac{2\pi i}{e^{\frac{1}{h}}}\right)$". (Zweiter Teil), p. 758, (Dritter Teil), p. 774. (Communicated by Prof. W. KAPTEYN).

Physiology. — *"Our equilibrium-organ"*. By Dr. D. J. HULSHOFF POL. (Communicated by Prof. C. WINKLER.)

(Communicated in the meeting of November 24, 1917).

When studying the functions of the cerebellum [1]) I always halted before the difficulty of interpretation of the observed phenomena. It is not easy to deduce whether they are dependent on the organ in itself, or resulted from the interruption of tracts in the cerebellum, which took origin in other parts of the central nervous system.

Therefore it seemed to me desirable, before continuing my investigations on the cerebellum, to trace in the first place the connection between those influences which are lying outside of this organ and the cerebellum itself.

As experiments on this subject have often been made, it was clear that repetition of the former investigations would not bring a nearer solution of this problem.

Therefore I resolved not to start by experimenting on animals, but to examine the suffering people and especially to pay full attention to ataxia.

As this phenomenon is often observed by disturbances of the central nervous system and as it is known, that according to the illness, also the decomposition of the movement can show a different type, I thought, that perhaps it could be possible by putting together the different types to get a more distinct insight into the nosology of the cerebellum.

Now it is in general accepted that cerebellar ataxia is caused by a disturbance in efferent paths, and thus could be a motor disturbance, but as this kind of conducting fibres is not well imaginable without afferent tracts, one may accept that these too, interrupted in the cerebellum, will show disturbance in movement.

As my former investigations were exclusively restricted to the motor functions, I thought it now wise to draw attention to the afferent tracts.

---

[1]) *Cerebellar ataxia.* Phsych. Neurol. bladen 1909 N°. 4.
Cerebellar functions in correlation to their localisation. Psych.-Neurol. bl. 1915 N°. 3.

Of the sensory and sensorial stimuli which reach the cerebellum along centripetally conducting paths and which are well-known to us, we may mention in the first place the functions of the deep sensation and of the sense for the muscle tone and the equilibrium.

The first pass with the posterior nerve-roots into spinal cord and run partially uncrossed, with the Column of Clark as mid-station, to the tractus spino-cerebellaris dorsalis (FOVILLE-FLECHSIG), while another part, with the area nuclei intermedii as mid-station, run also for more than the greater part uncrossed to the tractus spino-cerebellaris ventralis (GOWER), while a smaller part goes to the same column of the crossed side. This bundle is therefore partially composed of crossed, partially of uncrossed fibres.

The tracts of FLECHSIG and GOWER lie in a long but narrow strip at the lateral edge of the spinal cord and run centripetally.

The tract of FLECHSIG goes through the restiforme body and the inferior brachium conjunctivum towards the vermis of the cerebellum, without coming in contact with the dentate nucleus.

The tract of GOWER does not pass into the restiforme body after having reached the medulla oblongata, but runs on in longitudinal direction. On the level of the nervus trigeminus it bends round in latero-dorsal direction and passes along the brachium conjunctivum cerebelli into the vermis superior and the nuclei tecti cerebelli.

From the posterior columns of the spinal cord however, there are along other paths also tracts connected with the cerebellum, e.g., through the nuclei of GOLL and BURDACH, along the fibrae arcuatae externae and anteriores to the restiforme body and from here to the cerebellum.

As the influence of these latter fibres is far from known, I will leave them in the further discussion out of the question.

Now the experiments of MARBURG and BING [1]) have taught, that the lesion of the spino-cerebellar tracts provokes a very serious disturbance of the statotonus. Partial or total destruction of these bundles from the entrance in the spinal-cord to the cerebellum, will therefore show disturbance of the equilibrium.

If the connection between the tracts of FLECHSIG and GOWER with the cerebellum are well-known, this is less the case with those between the vestibular organ and the cerebellum.

LANGELAAN [2]) writes that the end-arborisation in the oblongata of

---

[1]) EDINGER. Zeitschr. f. Nerv. Heilk. V. 45, 1912 p. 303.

[2]) J. W. LANGELAAN. Bouw van het centrale zenuwstelsel. Amsterdam. VERSLUYS. 1910.

the vestibular nerve is T-shaped, of which the ascending fibres continue into the cerebellum. They unite into bundles, between which is found gray matter, belonging to the nucleus of DEITERS. These bundles form the greater part of the corpus juxta-restiforme and pass with the fibres of the corpus restiforme into the cerebellum, where they end in the dentate nucleus and in the nucleus tecti. JELGERSMA [1]) too points out (p. 217) the fibres of the nervus vestibularis as thick bundles running through and along the bracchium conjunctivum inferius towards the cerebellum, being everywhere visible as distinct bundles. WINKLER is of a different opinion and thinks from sections, which he possesses, he can make out that the nervus vestibularis does not stand in direct connection with the cerebellum, but that the fibres all end in the corpus juxta-restiforme, around cells of the nucleus of DEITERS, the nucleus triangularis, of the proper nucleus of the radix descendens N. VIII. Cells of middle size, lying in the regions of these nuclei, carry the impulses through the curpus juxta-restiforme towards the cerebellum. It does not receive direct nerve-roots of the N. VIII.

In this case therefore the connection has to take place by means of an interjacent link.

As to the physiological function, the investigators of this region are almost of the same opinion, that the vestibular organ will be *an organ for the muscle tone and for the equilibrium.*

Its great importance for our equilibrium has gradually and regularly come to the foreground, even so, that GOLTZ made a sixth organ of it. [2])

We may say, therefore, recapitulating in short the above mentioned that from the spinal cord as well as from the vestibular organ, strong tracts run to the cerebellum and that interruption of these disturbs the equilibrium.

In my investigations I thought I was allowed to start from the standpoint, even although from both organs paths go to the cerebellum, yet the difference between the stimuli which they conduct, is so great, that it could perhaps show me the way in the intricate mass of the cerebellar phenomena.

Thus, as the spinal path, taking its origin in the peripheral nervous system, enters the spinal cord through the posterior nerve roots and i.a. as the thick posterior fibres of the roots send their collateral fibres to the column of CLARK and the area nuclei intermedii, I thought I could best study the disturbance in the equilibrium when

[1]) G. JELGERSMA. De functie van het cerebellum. Psych. en Neur. bl. 1915.

[2]) H. ZWAARDEMAKER. Physiologie. De Erven F. BOHN. Haarlem 1915. bl. 286.

the centripetal impulses along the tractus spino-cerebellares were fallen away, as happens in cases, affecting the posterior funiculi, e.g. in tabes dorsalis.

For disturbances of the vestibular organ I had best limit myself to those cases, in which the equilibrium organ had lost its functions, inter alia after scarlatina, cerebro-spinal meningitis etc.

In working out these investigations I tried in the first place to devise a scheme, which could serve as well in lesions of the posterior funiculi, as in those of the vestibular organ and in alterations of the cerebellum.

I thought I had found one in letting the patients perform walking-tests, which were registered on paper. For this purpose a line was drawn in the middle of large pieces of paper, on which the patients walked after blacking their footsoles.

When in this paper I leave the results, which I found in cerebellar alterations out of account, and when I limit myself to those which are correlated to lesions of the posterior funiculi and the vestibular organ, I find what follows:

1. If a patient, who suffers from tabes in a rather far advanced state, walks on the paper, then the reproduction of figure 1 [1]) appears:

The patient tries to fulfil the task of walking on the line (foot print 1—2 and 3), but sways to and fro, as in the ROMBERG syndrome and he is obliged to put down the right foot lateralwards, (4). Still worse the swaying becomes in the following right footstep (6) when he replaces the foot three times to keep his equilibrium.

The deviation in the line of equilibrium is most distinct with the footsteps 8 to 11, which he had to put down close to each other and during which it was impossible to him to remain on the line. He therefore leaves off trying it and walks on rather well along a broad gait-path.

2. When that which is ordered sub I, is repeated, but with eyes shut, then the deviations of the gaitpath are still more districtly visible.

3. If one lets the patient repeat the same walking exercises as sub 1 and 2, but allows him at the same time to touch our hands [2]), then one sees a gaitpath nearly as normal (fig. 2).

---

[1]) The cross on the photo indicates the moment when I thickened the contours of the foot prints with ink, because otherwise the footsteps do not come out well enough on the photograph.

[2]) While walking the patient, who is standing on the middle of the paper, stretches his arms to the left and to the right and lays his hands on the dorsal plane of those, which are tended to him from the side. The persons, who help to

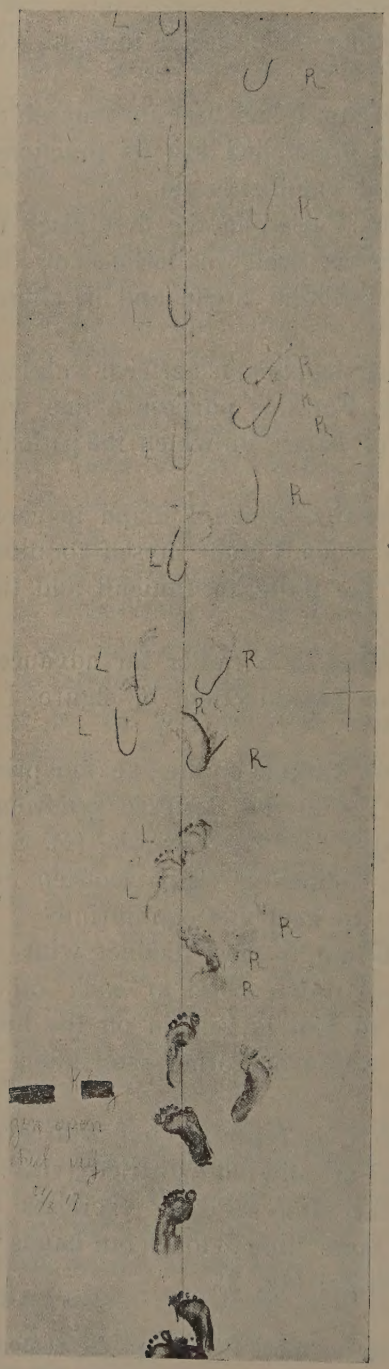


Fig 1.
Tabes no contact-sensation.

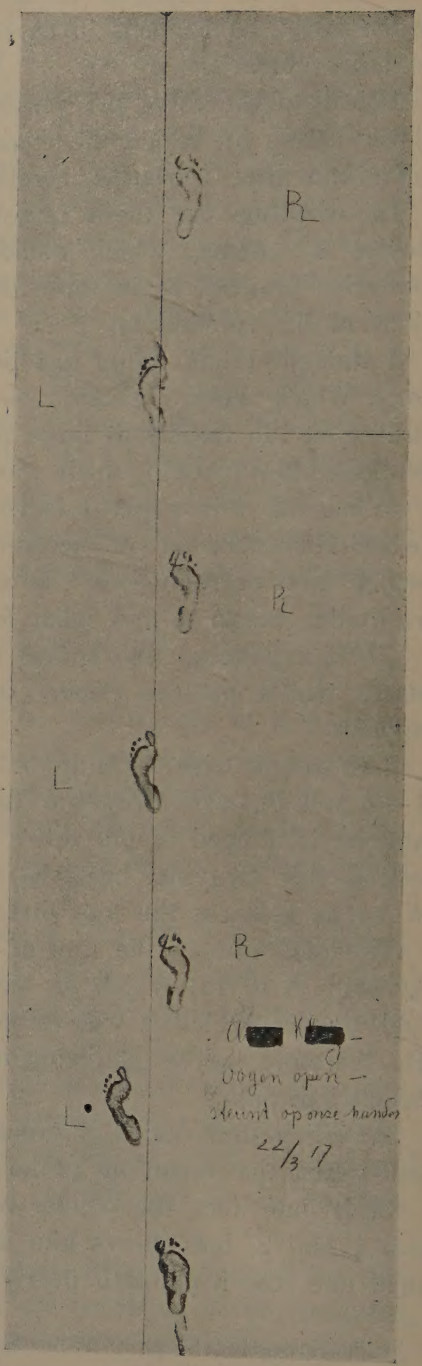


Fig. 2.
Tabes contact-sensation.

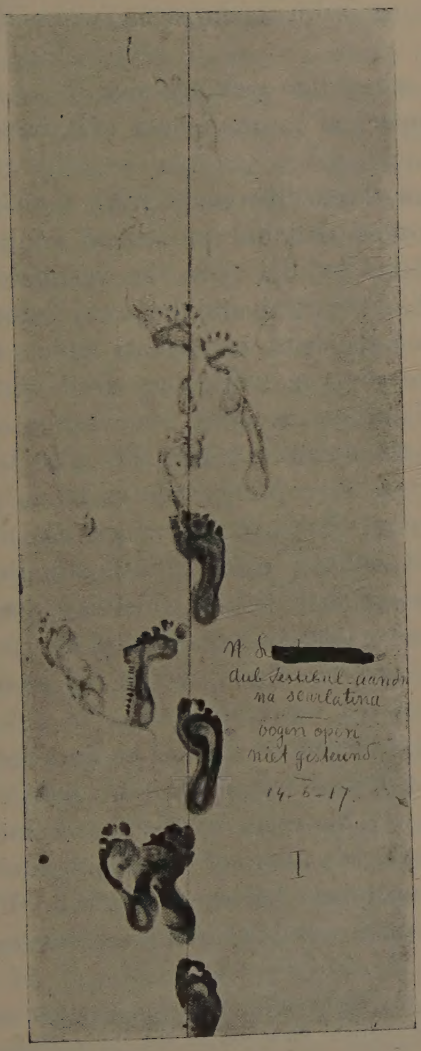


Fig. 3.
Vestibular affection no contact sensation.

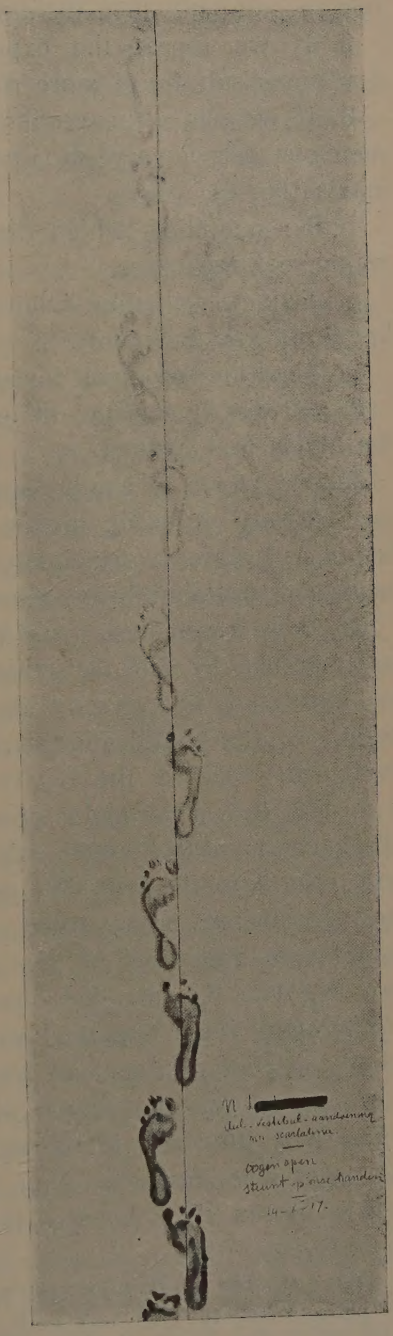


Fig. 4.
Vestibular affection contact-sensation.

4. If one reproduces of a patient, suffering from a vestibular inflection, a gait-path, then distinct deviations will be visible (fig. 3).

5. If we repeat the experiment, but with eyes shut, then the deviations sub 4 are more distinctly visible.

6. If we allow the vestibular suffering patient to touch our hands, then one gets a very important amelioration, even a nearly abolished ataxia (fig. 4).

If the symptoms which these patients show are put together, then we have three types:

a. distinct deviation while walking with open eyes, while the hands do not find contact.

b. increasing of these deviations, when the eyes are shut.

c. gait nearly normal, at least important improvement of it, when the hands find contact.

If one wants to comprehend these three differences well, then it is necessary to bring to the foreground that our movement equilibrium, as it were, is principally regulated by the eyes, the vestibular organ and the equilibrium sensation [1]) of the trunk and the lower limbs.

Of these three factors the eyes are the least important, which is easily tested by the fact that a person can walk very well with his eyes shut, but directly shows disturbances, when the vestibular organ or the equilibrium sensation are suffering. This can also be explained, because the last two factors give proprioceptive stimuli, according to the particular conception of SHERRINGTON, which do not affect consciousness, while the eyes convey exteroceptive stimuli, with regard to the gait. We could describe it best in this way, that the equilibrium is governed by the vestibular organ, also by the equilibrium sensation of the trunk and the limbs, while the eyes only regulate the intended direction of the movements.

Therefore when a patient suffering from tabes walks, he does not wholly dispose of the three above mentioned factors, but he walks, if we are allowed to express it thus, by his eyes, by the vestibular organ and the rests of the equilibrium sensation of the trunk and the legs. The result is, that the movements become uncertain. If such a person shuts his eyes, then the exteroceptive stimuli, moreover,

---

do this and *who walk along the edge of the paper*, are asked *not to support* the patient, but to give way as it were in vertical direction to the movements, which the patient makes. Their hands therefore have to balance too. The patient has no support, but only contact sensation with the persons who walk alongside of him, and by which his equilibrium sensation can orientate itself.

[1]) Equilibrium sensation has to be interpreted as an independent subdivision of that, which till now is brought together in conception of "deep-sensation".

fall out, because the eyes cannot give direction to the intended movements and therefore it is comprehensible that the uncertainty of the gait augments.

However on the other hand it can be said that as the equilibrium in particular is only regulated by proprioceptive stimuli, disturbances in gait will occur, when a part of those fall out, but the disturbances will be partially improved by the exteroceptive stimuli, which by means of the eyes can convey their stimuli.

The above mentioned is known, but it is most important, that when to such a tabes patient, either with opened, or shut eyes, *contact* is given through persons walking alongside of him, the gait greatly improves, even so, that the ataxia nearly altogether disappears.

As such a patient is *not supported*, but as he has only *contact* with persons walking alongside of him, I think it may be assumed that the equilibrium sensation of the upper limbs is put into action. The equilibrium-sensation orientates itself along this new path, brings in this way new afferent proprio-ceptive impulses towards the central nervous system and therefore it can better control the movements of the lower limbs. The equilibrium sensation of the arms, in the tabes patient, thus takes over the function of the equilibrium sensation of the lower limbs and trunk (this only of course when the tabes is present in the caudal part of the spinal cord), which for a great part has disappeared.

Therefore we can say the following:

a. a tabes patient walks by his eyes, by the vestibular organ and the *rests* of the equilibrium sensation of the trunk and the legs.

b. if such a patient has contact-sensation with persons walking alongside of him, then he *moreover* walks by the equilibrium sensation of the arms.

In case *a* he lacks afferent equilibrium impulses on a great scale and therefore he walks atactic, in case *b* the amount of these equilibrium-impulses is very considerably augmented and the ataxia therefore is improved, nay it even has entirely or almost disappeared.

If we have a patient suffering from a vestibular affection, then we see on the whole the same effect.

In this sort of patients the central nervous system too receives the exteroceptive impulses by way of the eyes, and moreover the proprioceptive ones of the trunk and the legs, through the posterior roots of the spinal cord, but none or only partially from the proprioceptive ones of the vestibular organ.

The result of this is, that the amount of equilibrium impulses is not sufficient, therefore the patient walks atactic.

For these cases too it is important, that when the patients, by means of their arms, receive equilibrium contact through persons walking at their side, then the ataxia either very importantly improves or it entirely is abolished.

Here too we see, that the equilibrium contact of the arms replaces totally, or for a very large amount the proprioceptive impulses from the vestibular organ.

Therefore we can say for this case:

a: a vestibular patient walks by his eyes, by the equilibrium sensation of the trunk and the legs, and the afferent-proprioceptive stimuli from the vestibular organ, which is left to him.

b: if such a patient has equilibrium contact with persons leading him, then he *moreover* walks by the equilibrium sensation of the arms.

In case *a* he lacks afferent equilibrium stimuli and the patient walks atactic. In case *b* the ataxia totally or partially disappears, because his lack is supplied.

Now the peculiarity of the results found is, in tabes as well as in vestibular affection, *that the equilibrium sensation of the arms can compensate the equilibrium sensation of the trunk and legs as well as the impulses from our vestibular organ.*

On account of this the question arises whether it is possible, that in our equilibrium different organs can replace each other.

This question deserves to be answered in the affirmative to a certain extent.

If e.g. we close the eyes of a person, who is then asked to walk straight on, there will be many, who deviate to the right or to the left. The reason of it will depend among other things on the fact, that the proprioceptive equilibrium stimuli, which arise from both the halves of the body, are not of the same strength; the result is that one half predominates and that the gait will not be totally straight. If we place, however, at a distance a person, who counts, then the blindfolded person will be able, guided by the sound, to walk straight on towards the counting person. The extero-ceptive stimuli, which pass from the ears towards the cerebrum, complete the others, through which the straight gait is made possible. The sense of hearing comes to the aid of the equilibrium sensation. It is also well-known that the eye sense can give direction to our movements.

It is comprehensible that as these two senses are already able to give assistance under normal circumstances to the equilibrium sensation, they can help the suffering person in yet higher degree after practice.

It is also a well-known fact that e.g. the ataxia in tabes patients, who can still walk straight with their eyes opened, comes to the

foreground, when the same movements are performed with eyes shut, or when they are walking in the dark and cannot make use of their eyes.

This help may be rather sufficient in light cases of tabes, but it will not be possible to totally improve the disturbance, if the illness has become of a rather serious nature. If we give to this kind of patients contact-sensation by the arms, there will yet be an important improvement. (Fig. 2).

Consequently from this follows as is moreover near at hand, that the equilibrium sensation of the arms, being of the same sort as of the trunk and limbs, compensates in reality, while the other senses can only correct to a certain degree.

I have pointed out in the preceding pages, that one finds the same facts back in lesions of the vestibular organ. Here too the ataxia improves, when the patient uses his eyes (Fig. 3), but here too one finds, that *when the eyes cannot sufficiently correct any more, the ataxia totally or nearly totally vanishes, when through the arms equilibrium-sensation is obtained with the surrounding world.* Here too we find, that the equilibrium sensation of the arms acts totally or nearly totally compensatory. (Fig. 4).

The question arises how we can explain this.

I think that I may except as easy to comprehend, *that the proprio-ceptive stimuli of the equilibrium sensation of the trunk and the limbs and those of the vestibular organ are to be considered of the same sort.*

Referring to our equilibrium, no difference should be made between the afferent-proprioceptive stimuli, which from the vestibular organ are conducted to the central nervous system and those which come there from the trunk and the limbs. It is a large system of equilibrium fibres that is spread over our whole body and its aim is to regulate the equilibrium.

The sixth sense, the one for our equilibrium, has therefore not only to be looked for in the vestibular organ, but it is, as I explained, spread over the whole of our body. The vestibular organ is but a part of it.

Now probably one might ask, why does that organ form a whole while in the other equilibrium paths very little independency is found. The reason for this, according to my view, has to be found in the extraordinary relation of the head, in comparison to the rest of the body.

To make this clear, one has to keep in mind, that the equilibrium sensation of the different parts of our body is not everywhere the same, or otherwise expressed, is no teverywhere equivalent. E.g. the equilibrium sensation for the trunk, which can only move

moderately, is but little developed. For the lower limbs it must be already higher, because the movements which they perform are already much more complicated.

In yet higher degree this is the case with our arms. So it is known, that as soon as our equilibrium gets in any danger, we do not only immediately put our arms into function, but that we trust even more to our arms, which are weaker concerning our muscle strength than to our much more muscular legs. If the equilibrium sensation of our arms has already reached a high degree, this will be yet more so with our head, which is above all designed to bring to our knowledge our attitude in space. It speaks for itself that without this knowledge no equilibrium is possible. Moreover, with exception of the lower jaw, the different parts of the head are not linked together by joints, but they are tightly grown together. That this puts the head into an extraordinary relation is comprehensible. The trunk and the limbs are in opposition to the head composed of movable parts, which are joined together by tendons and sinews. All the changes in attitude, therefore also those which are of importance for our equilibrium, come to our consciousness. This is not the case with our head.

By means of the neck and all that is connected with it, it can, however, fix its own *posture* with relation to our body, but it will not be of any use for the determination of the equilibrium.

The equilibrium organ of the head has, on account of what was above reported, to be not only much more highly developed, but it must be differently composed from the equilibrium paths for the trunk and the limbs. Therefore too the vestibular organ is built up in a different way. As the joints in our head, necessary for the equilibrium, are all missing, it is most probable, that the statoliths through movements during the changes of attitude stimulate the equilibrium fibres aud thus put into action the necessary afferent-proprioceptive impulses. The central nervous system is immediately warned of any danger that threatens our equilibrium and can take the necessary steps against it.

CONCLUSIONS.

a. The sixth equilibrium sense is not placed in the vestibular organ only, but has its tracts spread over the whole body.

b. The vestibular organ is, as far as it refers to our equilibrium only to be considered as a part of the equilibrium sense.

c. The different parts of this sense can compensate each other reciprocally.

Physiology. — "*Cerebellar ataxia as disturbance of the equilibrium-sensation.*" By Dr. D. J. HULSHOFF POL. (Communicated by Prof. C. WINKLER.)

(Communicated in the meeting of Jan. 26, 1918).

In a former paper[1]) I explained, that the ataxia which a patient, suffering from tabes, shows, while walking, is nearly quite abolished when he has by means of his hands contact-sensation with the persons walking to his right as well as to his left side.

The reason for this improved walking has to be looked for in the fact, that as long as the affliction resides in the lower part of the spinal cord, it enables him to make use of the equilibrium-sensation of the upper limbs (afferent-propriaceptive stimuli), and in this way he can orientate himself better in space.

If this latter happens to be, then all the efferent impulses, necessary for the regulated movements, can run down along the motor paths to the trunk and the lower limbs, through which the ataxia becomes abolished, respectively ameliorated.

I expressed myself thus, that where an ordinary person walks by his eyes and the equilibrium-sensation of the vestibular apparatus, the trunk and the lower limbs, a tabes-patient does it by the equilibrium-sensation of the vestibular apparatus and the *rests* of that which is still left in the trunk and lower limbs.

If such a person therefore has contact-sensation with people walking alongside of him, he moreover walks by the equilibrium-sensation of the upper limbs.

If one examines, not a tabes-patient, but a sufferer from the vestibular organ, then, as I wrote, such a patient will walk by his eyes, the equilibrium sensation of the trunk and legs and the *rests* of the equilibrium impulses which are obtained through the vestibular apparatus. If one therefore gives to such a patient contact sensation through persons walking alongside of him, then it also appears that he is enabled to walk normally again or at least nearly normally.

As the ataxia, which both these patients show, are both almost totally abolished through the same influence, viz. contact sensation

---

1) Our equilibrium-organ. These Proceedings p. 626.

by means of the upper limbs, I thought I had to accept the involved afferent-proprioceptive impulses in those cases as being of the same sort.

The vestibular apparatus therefore has to be considered, *as far as our walking function is concerned*, as a modified and higher developed organ for the equilibrium-sensation of the head.

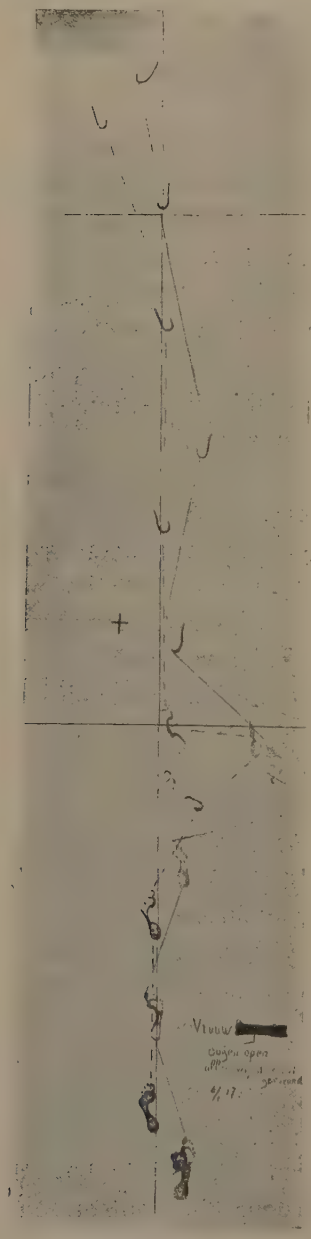


Fig. 1.

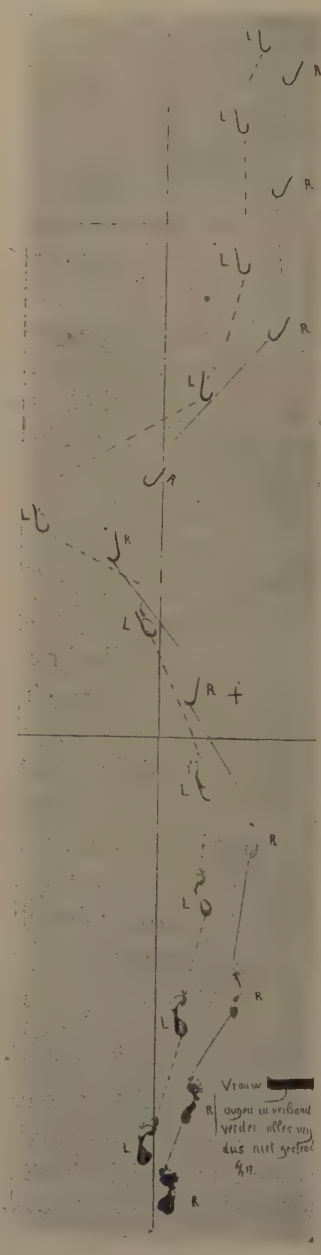


Fig. 2.

It would take me too long in this paper to enter more in particular into this hypothesis; I refer the interested reader to the original communication. The one thing to which I will draw attention, is, that according to this view, considering our equilibrium, there should be made no principal difference between the afferent-proprioceptive stimuli which are conducted from the cerebellar tracts of the lateral column and those from the vestibular apparatus to the cerebellum. All these stimuli are related to the equilibrium-sensation and therefore regulate our gait. These different afferent tracts thus form a whole and they are to be considered as a subdivision of the same equilibrium system.

Now it is important to trace, whether they change, and if so, what alterations these afferent stimuli undergo, when they arrive in the cerebellum.

If, investigating this, one makes patients, suffering from e. g. cerebellar tumors, perform the same walking-tests, as I did with sufferers from tabes or vestibular affliction, then the results prove to be totally the same.

For instance I will report the following case, in which during life the diagnosis was made of a tumor, which was located at the left side, where it pressed as well on the cerebellum as on the nervus octavus. During the operation and also post-mortem the diagnosis could be confirmed.

If one had the patient walk totally unsupported with open eyes, then the part of her gait was obtained as reproduced in fig. 1. The ataxia is distinctly visible. If her eyes were shut, then the path of her gait became as in fig. 2.

When comparing the two the ataxia proves to be considerably augmented. This is comprehensible, because in the latter case the afferent-extero-ceptive stimuli from the eyes are missing. The difference between these two paths of gait results from the influence of the eyes on the performed movements.

It is important to point out once more (see preceding paper) that, when by abnormalities of the static organ a second organ comes to his aid, the latter only can partially replace the deficiencies of the former, if these are of rather severe nature.

This is distinctly visible in fig. 1, because notwithstanding the aid of the eyes, the ataxia however is far from gone.

If we now let such a patient walk quite free with eyes open, but giving her contact sensation through persons next to her, then *although also the neck-equilibrium-impulses are shut out* by means of a bandage, the gait path in fig. 3 is reproduced, in which the

ataxia proves to be totally gone. The equilibrium sensation which is augmented through the contact-sensation of the arms is enabled to compensate all that is missing. What the eyes could not do, the afferent-proprioceptive stimuli of the upper limbs could do. They compensated, thus abolishing the ataxia.

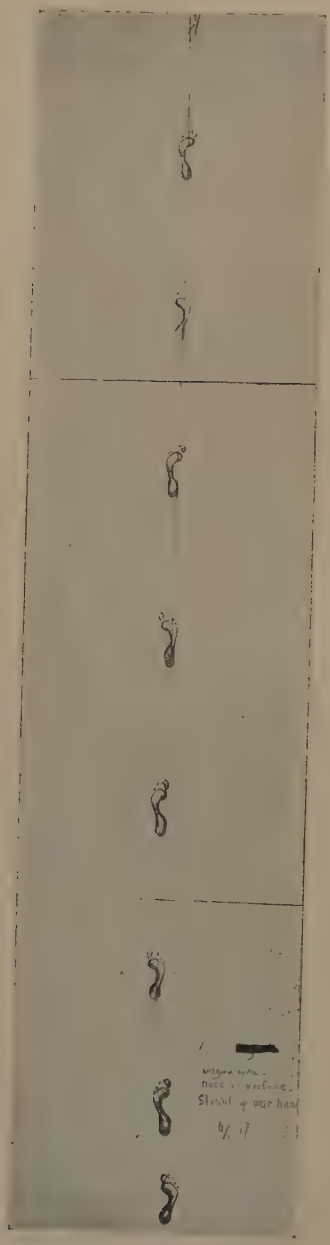


Fig. 3.

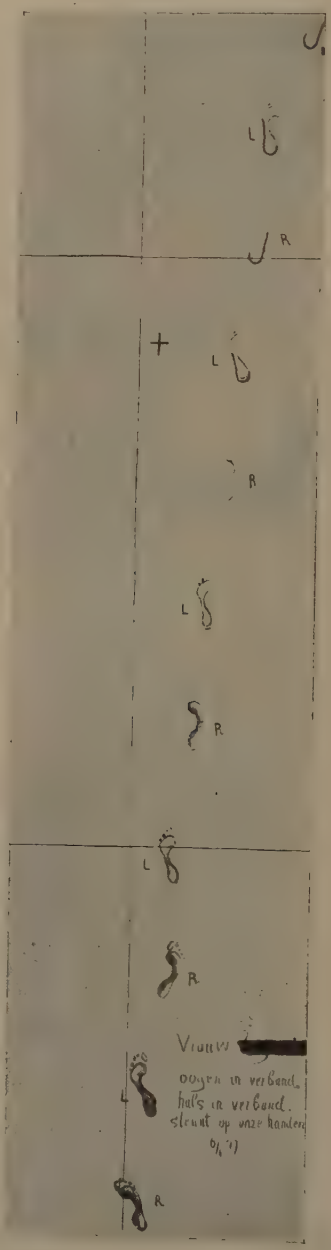


Fig. 4.

As one could oppose against this important amelioration that the result had been due for the greater part to the help of the eyes, I repeated the test in precisely the same way, but now with eyes closed. The produced gait path is found in fig. 4. Now too no sign of ataxia is seen. The only deviation is, that the patient walks to the right instead of walking straight on. As this deviation takes place gradually and the gait-path remains straight, the circus gait must be excluded; it seems to me, that in this case one cannot speak of a deviation to the right. It is most probable that the patient did not stand straight in front of the line, but somewhat in oblique direction and therefore walked in that way.

The 4th fig. proves that the contact sensation through the arms is yet able to abolish the ataxia, even if the patient misses the afferent-extero-ceptive impulses from the eyes.

One thus sees here exactly the same phenomenon as with the ataxia in tabes and in vestibular affliction.

Therefore it lies at hand to assume that cerebellar sensory ataxia arises when in the cerebellum the equilibrium paths are being interrupted which from the spinal cord and the nervus octavus pass into it.

This would confirm JELGERSMA's [1]) view (pg. 217): "The supposition that the cerebellum is a central place of innervation for both these organs [2]), is therefore probable."

Should my view be right, then cerebellar sensory ataxia will occur, when the process of the disease arises in those parts of the cerebellum, in which the equilibrium-paths pass.

In connection with the above a few questions arise.

The first is whether the cerebellum exclusively dominates the equilibrium.

This question is answered in negative sense by many investigators, because e.g. experiments on animals have taught that experimentally obtained cerebellar ataxia can totally pass away after a certain time, which proves that the afferent equilibrium impulses can arrive at the cerebrum also along other paths.

The second question is whether the cerebellar ataxia always shows the same image.

Should I, as regards the experiments on animals, confine myself to the well-known investigations of LUCIANI [3]), then he too made a

---

[1]) JELGERSMA, G. The function of the cerebellum. Psych. Neur. bl. 1915.

[2]) Meant are: tonus- and equilibrium organ and the deep sensation.

[3]) LUCIANI, L. Das Kleinhirn. Georg. Thieme. Leipzig 1893.

difference between the ordinary atactive movements and those which were indicated by him as "dysmetria".

When one wishes to see these differences reproduced distinctly I refer to my photographs in my communication on "cerebellar ataxia". Plate VIII, X to XIV [1]).

In man, where the phenomenon cannot be localized as sure as in animal-experiment, a conclusion is drawn with more difficulty. Yet I will quote some writers, who showed that the deviations in gait do not always give the same aspect.

JELGERSMA (l. c. p. 227) e. g. writes that the occurrence of the cerebellar ataxia is due to the fact that the trunk sways over the legs: "a gait, which greatly resembles the walking of drunken men."

He only describes one type of deviation of the gait. OPPENHEIM, however, differentiates in his hand-book two forms of cerebellar ataxia, 1. "auf (Schwindel und) Gleichgewichtsstörung beruhende, "die grosse Aehnlichkeit mit der Gangweise des Betrunkenen "zeigt," and 2. "eine auf Bewegungsataxie beruhende. Patient "geht breitbeinig und stampfend, aber ohne dass ein übermässiges "Schleudern eintritt .... Eine scharfe Unterscheidung dieser Gehstörung "von der spinalataktischen ist wohl nur möglich, wenn sich die unter "1e beschriebene Abart mit ihr verbindet."

OPPENHEIM therefore thinks the cerebella ataxia e.g. also dependent on the spinal cord.

DÉGÉRINE on the other hand writes in his work "Traité de Pathologie générale" 1901, on page 643 "*b.* Ataxie labyrinthique. Les affections "de l'oreille interne produisent quelquefois des troubles de la marche "et de l'équilibre, qui ressemblent jusqu'a un certain point aux "disordres de l'ataxie cerebelleuse." DÉGÉRINE points out in this case the connection of the cerebellar ataxia with the phenomena which are found in vestibular disturbances.

Now the question arises to interpret the difference in the atactic aspect.

In my former communication I explained that even although the equilibrium-impulses, originating from the vestibular organ may be considered as to belong to the same which come from the spinal cord, yet there exists a great difference in their results, owing to the higher development of that organ.

The consequence of this is, that according to the afferent equilibrium path suffering more in the one case than in the other, the complex of atactic symptoms will also appear differently.

If the spinal tracts have suffered most, then the type as described

---

[1]) Psych. Neur. Bl. 1909 N°. 4.

by OPPENHEIM will be found in general, viz. his second form of cerebellar ataxia.

Should the vestibular tract be hurt, then the description of DÉGÉRINE comes right. If there is an interruption of both the paths or if the disturbance of the paths from the right and from the left side commences more or less simultaneously, then perhaps the gait of the drunken man will become more prominent.

If the results of the animal experiment are compared with those found in man, then we may say that the cerebellar ataxia does not always show the same aspect and that this can be explained by the fact, that the cerebellum possesses more than one afferent tract, whose interruption causes disturbance in its course and that according to the suffering of the one or the other or more paths, the aspect will change.

The *third* question is to explain in cerebellar disturbance the fact how the equilibrium sensation of the upper limbs can totally compensate the ataxia, because one may accept that its equilibrium paths, just as those of the lower limbs, pass into the cerebellum and therefore will also be broken off by the process of the illness.

I must acknowledge that I cannot very well give an explanation of this fact, if not the possibility should be accepted of a better connection of the equilibrium sensation of the arms (apart from the cerebellum) with the cerebrum, than is the case with the legs. Later experiments will have to give a decision on this point.

CONCLUSION.

1. By interruption in the cerebellum of afferent cerebellar tracts, originating from the vestibular organ and the tractus spino-cerebellares, ataxia appears.

2. According as these tracts suffer more or less, whether alone, or together, the aspect of the cerebellar ataxia will present a different type.

Physics. — *"On the Heat of Dissociation of Di-atomic Gases in Connection with the Increased Valency-Attractions $\sqrt{A}$ of the Free Atoms"*. By Dr. J. J. VAN LAAR. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of October 27, 1918).

Introduction.

1. In a series of Treatises in these Proceedings [1]) on the additivity of the values of b and $\sqrt{a}$ of the equation of state, and on the fundamental values of these quantities for different elements in connection with the periodic system, I determined the following values of $b.10^5$ and $\sqrt{a}.10^3$, expressed in so-called "normal" units.

The values of $b_k.10^5$ (per Gr. atom), determined up to now, are found collected in Table I.

TABLE I. (Values of $b_k.10^5$).

H = 59 (34; 14)						He = 105
Li = 145		C = 100 (75)	N = 85 (60)	O = 70 (50)	F = 55	Ne = 76
Na = 270		Si = 155	P = 144	S = 125	Cl = 110 à 115	Ar = 144
K = 480		Ti = 180				
Cu = 110		Ge = 210	As = 195	Se = 180	Br = 165	Kr = 177
Rb = 580		Zr = 235				
Ag = 150		Sn = 265	Sb = 250	Te = 235	I = 220	X = 228
Cs = 710		Ce = 290				
Au = 150	Hg = 150	Pb = 320 à 375	Bi = 305			Nt = 277
		Th = 400				

It will be remembered that these fundamental values present a great regularity. Starting from the carbon group the decrease in every

[1]) These Proceed. **18**, 1220 and 1235; **19**, 2, 287 and 295 (1916); **20**, 138, 492 and 505 (1917). Cf. also Journ. de Chim. Phys. **14**, 3 (1916), and Zeitschr. f. anorg. u. allg Chem. **104**, 56—156 (1918).

horizontal row of the periodic system towards the right is 15 *units*, in every vertical row the increase downward 55 *units*. The elements of the valence-less Helium group evidently fit in with the elements of the preceding Halogen group in a natural way.

In this Helium group the values of b_k are directly calculated from those of T_k and p_k; also the value 59 for H from that of H_2 [1]). The other values have been calculated from the *compounds* of the different elements (for Cl the value 115 seems to be more satisfactory than 110). From T_k and p_k we can directly calculate for N, P, O, Cl from N_2, P_4, O_2 and Cl_2: $b = 86$, 135, 71 and 113 to 125 (according as for Cl_2 the data of DEWAR or the more recent ones of PELLATON are used).

All the values of b can now be built up *additively* by the aid of these fundamental values for the most divergent compounds. In this also the "condensed" values 34 and 14 are valid for H; for C, N, and O the condensed values 75, resp. 60 and 50. The rules holding for this are found in the two cited principal papers of 1916 in These Proceedings and in the Journ. de Ch. Ph.

No exceptions have been found to this additive law, those that still existed for a few organic substances (e.g. the *amines*) have now all disappeared, owing to the later determinations of T_k and p_k by BERTHOUD [2]) at Neuchâtel, undertaken expressly in connection with these deviations. The critical pressures determined earlier by VINCENT and CHAPPUIS appeared to be all faulty to a high degree [3]). It is to be foreseen that this will also be the case with other earlier determinations.

We will also state that the values of b for H, Li, Na, K, Rb and Cs are to each other in the ratio $^1/_2 : 1^1/_4 : 2^1/_8 : 4 : 5 : 6$.

The values of b found are *entirely independent* of the state in which the atom is: whether as *free* atom e.g. in metallic tin, or as part of a molecule as in $SnCl_4$. It will presently appear that this is *not* the case with the fundamental values of $\sqrt{a}$.

2. For the values of $10^2 \sqrt{a_k}$ per Gr. atom, again expressed in "normal" units, were found up to now the values recorded in table

---

[1]) Of course the latest data were used for this; for H_2 and Ne the values found very recently by K. ONNES, CROMMELIN and CATH.

[2]) Journ. de Chim. Phys. **15**, 3 (1917).

[3]) Thus p_k appeared to be $= 55{,}5$ for $NH_2.C_2H_5$, whereas V. and Ch. found 66; for $NH(C_2H_5)_2$ the value 36.6 was found, against 40 by V. and Ch.; etc. Also for C_3H_7Cl BERTHOUD found 45,2, whereas V. and Ch. found 49. Errors therefore from 10 to 20% !

II; the values printed in bold type refer to the *increased* values of the *free atoms*, the others to those which are found in *compounds*: hence the *rest-values*.

TABLE II. (Values of $\sqrt{a_k} \cdot 10^2$).

H = 1,1 (3,2; 1,6)						He = 0,8
Li = ? **23**		C = 3,1 **32**	N = 2,9	O = 2,7	F = 2,9	Ne = 2,1
Na = ? **27**		Si = ? **34**	P = 6,4	S = 6,3	Cl = 5,4	Ar = 5,2
K = ? **33**		Ti = ? **35**				
		Ge = ? **36**	As = 7	Se = 7	Br = 6,9	Kr = 6,9
Rb = ? **35**		Zr = ? **37**				
		Sn = ? **38**	Sb = 8,9 **33**	Te = 9	I = 8,8	X = 9,1
Cs = ? **38**		Ce = ? **39**				
	Hg = 10,7	Pb = ? **40**	Bi = 11 **36**			Nt = 11,5
		Th = ? **41**				

Not from the fundamental values of a can the values of a of all possible compounds be built up additively — but those of $\sqrt{a}$ from the fundamental values of $\sqrt{a}$ given in the above table. Here too no important deviations were found; those for the amines have again disappeared for the greater part through the later determinations of BERTHOUD (loc. cit.).

The values of $\sqrt{a_k}$ for the noble gases (calculated directly from those of T_k and p_k) fit in again perfectly with the values holding for every horizontal row. It may be said of them that (starting from the carbon group) the values of $\sqrt{a_k}$ in *compounds* are *about the same* for every horizontal row of the periodic system, and can be represented in approximation by the whole numbers 1. 3. 5. 7. 9. 11 from the 0. to the 5. row (inclusive).

The values of $\sqrt{a_k}$ calculated for N, P, O, and Cl directly from N_2, P_4, O_2 and Cl_2 are resp. 2,6, 6,4, 2,6 and 5,4 to 5,8 (DEWAR or PELLATON).

For H in compounds only the values 3,2 and chiefly 1,6 hold. The value 1,1 calculated directly from T_k and p_k of H_2 is only found for H_2 itself.

3. However — what is self-evidently *not* found for b — it should be taken into account that for the fundamental values of $\sqrt{a}$ the indicated attractions can be entirely or partly *destroyed* by an interception of the rays of attraction. Thus the value of $\sqrt{a}$ of a central atom surrounded on all sides by atoms or atom groups — as e.g. C in CH_4, C_2H_6 etc., Si in $SiCl_4$, Sn and Ge in $SnCl_4$ and $GeCl_4$, N and P in NH_3 and PH_3 — is found $= 0$ everywhere, so that these enclosed atoms do not exert *any attractive action towards the outside* [1]).

But as soon as the C-atom gets *partly free* again, by *double bindings* e.g. as in C_2H_4 (only imagine the position of the C-atom stereo-chemical), the value of $\sqrt{a_k}$ rises immediately from 0 to 1,55, hence half the fundamental value 3,1. And when the C-atom gets *entirely free*, for *triple bindings* e.g. as in C_2H_2, the full value 3,1 is accordingly at once found.

This rule holds everywhere. An interesting example is the *iso-amylene*. Here there are three singly bound and two doubly bound C-atoms. Hence the value $3 \times 0 + 2 \times 1{,}55 + 10 \times 1{,}6 = 19{,}1$ is calculated for $\sqrt{a_k}$. From T_k and p_k 19,2 was found. Indeed, an exceedingly remarkable confirmation of the rule.

The same thing holds for benzene and naphtalene. But for the singly bound C-atoms in the *substitution groups* the old value 0 is again valid. Thus for Tolnoene $= C_6H_5.CH_3$ the value of $\sqrt{a_k}$ is $= 6 \times 1{,}55 + 1 \times 0 + 8 \times 1{,}6 = 22{,}1$, while 22,2 is found; for O-Xylene $= C_6H_4(CH_3)_2$ we calculate $6 \times 1{,}55 + 2 \times 0 + 10 \times 1{,}6 = 25{,}3$, quite identical with the value 25,3 found from T_k and p_k. We might add numerous others to these examples, but we refer for this to the earlier principal papers.

We have now reached the *main question*: what happens when the atoms no longer occur in compounds, as N in N_2, Cl in Cl_2, etc. — but can occur *entirely free*, as for the *metals*, or for the *free atoms* Cl, Br and I in Cl_2, Br_2, I_2, which get decomposed at

[1]) Accordingly in consequence of this (see the table) we were not able to give the values of $\sqrt{a_k}$ for Si, Ti, Ge and Sn in compounds, though compounds of them are known, of which T_k and p_k have been determined. But it is exactly in *these* compounds ($SiCl_4$, $GeCl_4$ etc.) that the attraction of the central atom Si, Ti, Ge, Sn is eliminated.

high temperatures, or for H, N and O in the gases H_2, N_2, O_2, which also dissociate at very high temperatures?

Then — and this is perhaps the most remarkable of our results, which is fully confirmed by what follows presently with respect to the heats of dissociation — the values of $\sqrt{a}$ rise at once to the greatly increased values from **30** to **40** (instead of from 1 to 11). These increased values are therefore the real *valency-attractions*, whereas the values found in compounds represent only the so-called *rest-values*: what still remains for action outside, after the chief valencies are saturate, and have therefore become inactive towards the outside.

That for the elements of the helium group only the ordinary rest values are found in spite of their atomistic behaviour is of course owing to this that these elements are *valence-less*.

For arsenic only a partially increased value was found, pointing to a dissociation of As_4 at T_k to an amount of 20%, whereas phosphorus appeared to be still perfectly normal $= P_4$ at T_k. For Se and Te we found amounts of dissociation (always at T_k) of 30% resp. 80%. In the halogen group only a very slight degree of dissociation (5 and 10%) was observed for Br_2 and I_2. As the atoms occur more and more as *free atoms*, the *metal character* asserts itself more and more in a group: As → Sb → Bi; Se → Te; etc.

Besides the rise from 0 to 1.55 and 3.1 for *carbon* — according as the C-atom is entirely or only partially shadowed by surrounding atoms or atom groups, to which it is *bound* — another rise of $\sqrt{a}$ takes place, up to 32, the ten-fold value, when the C-atom has also got released from these bindings, and can occur *quite independent* as *atomistic* carbon. Hence the enormously high value of the critical temperature, viz. 6500 abs., which would only have amounted to 120° abs. in the case of C_2, as is easy to calculate. The carbon would have become comparable with N_2, O_2 etc., whereas it is now on a line with a metal that is exceedingly difficult to melt.

For *Tellurium* the peculiarity is still found that the normal rest-attraction 9, which is among others found for $TeCl_4$ [1]), has already

[1]) That for $TeCl_4$ the central Te-atom does not exert an intercepting influence like C in CCl_4, Ge in $GeCl_4$, etc. is owing to the configuration of the molecule. While for CCl_4 the four chlorine atoms are regularly situated in the *space* round the C-atom (in the direction of the four angular points of a tetrahedron), it should be supposed that for $TeCl_4$ the Cl atoms lie in one plane round the central Te-atom. Nor do we find a complete intercepting influence for PCl_3 and $POCl_3$, AsH_3 and $AsCl_3$, SbH_3 and $SbCl_3$, which for these substances is probably owing to the comparatively greater extension of the central atom. For $SbCl_3$ we find even again the full rest value 9 (Cf. the cited papers in these Proceedings).

risen to 13 for $TeCl_2$ — where two valencies have been liberated. For entirely free metallic Te $\sqrt{a}$ will rise still further, namely to about 30.

What consequences this behaviour of molecules and atoms with regard to the attraction exercised by them can have for the properties of many substances (volatility, surface tension, etc.), has been set forth at length by me elsewhere. [1])

4. The heat of dissociation of di-atomic gases.

After these introductory remarks we may now proceed to the real subject of this Paper, viz. the calculation of the *heats of dissociation* Q from the values of the increased valency attractions $\sqrt{A}$, which were mentioned above, and which are recorded in table II (the values printed in bold type under the elements), for so far as they are now known.

Let us take hydrogen as an example. For the internal energy of the not dissociated gas H_2 we may write:

$$E_{H_2} = (E_0 - Q_0) - \frac{a}{v} + c_{H_2} T, \quad . \quad . \quad . \quad . \quad . \quad (a)$$

in which E_0 represents the so-called *constant of Energy* of the *unbound* H-atoms, hence $E_0 - Q_0$ that of the *bound* H_2-atoms in H_2. Hence Q_0 is the absorbed (internal) heat of dissociation (in units of energy e.g.) in the dissociation $H_2 \rightarrow 2H$, at $T = 0$ — i.e. the chemical energy (at $T = 0$), which is liberated in the formation of 1 Gr. mol. H_2 from the separate H-atoms).

For the dissociated $H + H$ we have evidently:

$$2E_H = E_0 - \frac{4A}{v} + 2c_H T. \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

While, namely, for H_2 the quantity a represented the ordinary rest attraction (per Gr. mol.) between the *molecules*, A now represents the increased attraction of valency between the separate, now free *atoms* per Gr. atom, hence $4A$ per Gr. mol. $= 2$ Gr. atoms.

In the *ideal gasstate*, where v is very great, we shall simply have:

$$E_{H_2} = (E_0 - Q_0) + c_{H_2} T \quad ; \quad 2E_H = E_0 + 2c_H T,$$

and as these two quantities of energy will just differ the *total* (internal) heat of dissociation Q, we have then:

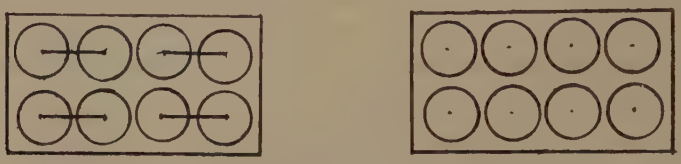
$$Q = 2E_H - E_{H_2} = Q_0 + (2c_H - c_{H_2})T, \quad . \quad . \quad . \quad . \quad (c)$$

the well-known expression for Q in the function of T, when c_H represents the limiting value of the specific heat at constant (large)

[1]) Cf. Chemisch Weekblad, Sept. 14, 1918, N°. 37 (p. 1124—1137).

volume of 1 Gr. atom H, and c_{H_2} the same quantity for H_2. As c_{H_2} is $= 5$, and $c_H = 3$ (in Gr. cal.), $Q = Q_0 + T'$ may be written in the case of $H_2 \rightarrow 2H$ for Q.

When, however, the volume becomes smaller and smaller, and the quantities $\frac{a}{v}$ and $\frac{A}{v}$ larger and larger, at last a (fictitious) volume will arise, in which the difference of the two energies has become $= 0$, in consequence of the fact that with respect to the internal energy it will then have become quite indifferent whether the atoms are separately present in that small space, or combined to molecules — i.e. when also the energies of translation do not differ, hence at $T' = 0$.



H_2 $\qquad\qquad$ $H + H$

For the difference $Q' = 2E'_H - E'_{H_2}$ we have in this case:

$$Q' = Q_0 - \frac{4A}{v_1} + \frac{a}{v_1} + (2c_H - c_{H_2})T', \quad . \quad . \quad . \quad . \quad . \quad (d)$$

in which v_1 represents the above mentioned small volume, which we shall have to define more closely. Now it follows immediately from (d) at $T' = 0$, in which case Q' must be $= 0$, that

$$Q_0 = \frac{4A}{v_1} - \frac{a}{v_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and this is the simple relation between the heat of dissociation Q_0 at $T' = 0$ and the two attractions A and a, which we have sought.

We must now determine the small volume v_1. This will evidently be of the order of the limiting volume, which the molecules themselves (see the above figure) occupy in the *natural* state in *unconstrained* condition, i.e. the volume expressed by b_g — and not e.g. the smaller volume b_1 in the liquid state, where the molecules will be *compressed* in consequence of the smaller space, and which therefore denotes a *constrained*, and no free, no natural condition [1]). Now b_g is about $= b_k$, so that we may put:

$$v_1 - b_g = \gamma b_k,$$

[1]) In this it has also been supposed that b_g is not $= 4m$, when m denotes the *real* volume of the molecules, but simply $= m$ itself. According to recent views the latter supposition is theoretically at least as well justified as the former $b_g = 4m$, which refers specially to collisions of *mathematical spheres*, and not of *real*

in which $\gamma(>1)$ will not be far from unity. When we put $4A = na$, (1) becomes:

$$Q_0 = \frac{(n-1)a_1}{v_1} = \frac{(n-1)\varepsilon a_k}{\gamma b_k},$$

in which we put a_1, the attraction in the small volume $v_1, = \varepsilon a_k$. In this $\varepsilon(>1)$ will differ somewhat more from unity than γ, so that ε will be $> \gamma$. For $a_k : b_k$ we may now write $\frac{27}{8\lambda} RT_k$, in which γ is only little smaller than 1, so that we finally get, when $\gamma\lambda : \varepsilon = \theta$ is put:

$$Q_0 = (n-1)\frac{27}{8\theta} RT_k, \quad \ldots \ldots \ldots \quad (1a)$$

which enables us to calculate Q_0, when A and a, and in consequence of which also n and T_k are known. But as we do not know the values of A for H_2, N_2, O_2, but as we on the other hand *do* know those of Q_0 in approximation, we shall follow the *reversed* course, and calculate n from Q_0. We can then see whether the values of $\sqrt{A}$ calculated in this way are really of the expected order of magnitude, compared with the already known values of $\sqrt{A}$ for elements of the periodic system lying near (cf. Table II). We then get:

$$n = 1 + \frac{4}{27}\theta\frac{Q_0}{T_k},$$

hence:

$$4A = a\left(1 + \frac{4}{27}\theta\frac{Q_0}{T_k}\right),$$

when 2, the value in Gr. cal., is put for R, which renders it possible to leave also Q_0 expressed in Gr. cal. The value of a refers to 1 Gr.mol. H_2. When therefore a' represents the value of one *gram-atom* (*these* values are recorded in table II), then $a = 4a'$, and we get finally:

$$\sqrt{A_k} = \sqrt{a'_k} \times \sqrt{1 + {}^4/_{27}\theta\, Q_0/T_k}, \quad \ldots \ldots \quad (2)$$

when we take A and a' both at the critical temperature. Accordingly these two quantities now duly refer to 1 Gr.atom. Under the root sign, however, Q_0 refers to 1 Gr.mol. according to the derivation of the formula. As ε will always be $> \gamma\lambda$, $\theta = \gamma\lambda : \varepsilon$ will be *somewhat smaller* than unity, also when we take into account that R is not

---

molecules (which can differ considerably from the spherical shape, e.g. the elongated molecules of the hydro-carbons). Moreover, when the real size of the molecules is calculated by another method, we also find values that are in concordance with b_k ($\backsim b_q$), calculated from T_k and p_k, and *not* with $^1/_4 b_k$. Cf. among others my Article in These Proc. of Oct. 1914 (Vol. **17**), especially p. 883 and the Note on the same page.

exactly $= 2$, but somewhat smaller. The values of $\sqrt{A_k}$ *calculated* from (2) will, therefore, be sooner *too large* than too small, when we shall take for the present $\theta = 1$ in what follows.

5. Calculation of the values of $\sqrt{A_k}$ from Q_0.

Let us begin with *hydrogen*. ISNARDY (1915) determined $Q_0 = 95000$ Gr. cal., whereas LANGMUIR (well-known method) gave 132000 Gr. cal. in 1912. But the latter found from 70000 to 80000 Gr. cal. in 1914.

We are therefore not very far from the truth, when for H_2 we put the value of Q_0 at round 90000 Gr.cal. With $T_k = 33{,}18$ (K. ONNES, CROMMELIN, CATH, 1917) $Q_0 : T_k$ becomes therefore $= 2700$, hence $^4/_{27} Q_0 : T_k = 402$. And as $\sqrt{a'_k} = 1{,}1$ (see table II), we get (all the values of $\sqrt{a'_k}$ and $\sqrt{A_k}$ are still to be multiplied by 10^{-2}):

$$\sqrt{A_k} = 1{,}1 \times \sqrt{403} = 2{,}08 = \mathbf{22},$$

which is in very good harmony with what we have found for Li (23) and Na (27) (see table II).

For *nitrogen* BRINER [1]) calculates $Q_0 =$ at least 150000 Gr. cal. from a single value given by LANGMUIR. This value of BRINER's seems too high to me considering the values for H_2 and for O_2, and in my opinion 130000 Gr. cal. will be nearer the truth. When we calculate $\sqrt{A_k}$ from the two values of Q_0, we get $^4/_{27}\ Q_0 : T_k = {}^4/_{27} \times \times \left(\begin{matrix}150000\\130000\end{matrix} : 126\right) = {}^4/_{27} \times \left(\begin{matrix}1190\\1032\end{matrix}\right) = 176$ to 153. And $\sqrt{a'_k}$ being $=$ $= 2{,}6$, we get:

$$\sqrt{A_k} = 2{,}6 \times \sqrt{\tfrac{177}{154}} = 34{,}6 \text{ or } 32{,}3 = \mathbf{35} \text{ or } \mathbf{32}.$$

As in connection with $C = 32$ also a value in the neighbourhood of 32 can be expected for N, the result is also here a confirmation of our formula (1) or (1*a*), the more so when we bear in mind that the result may possibly be slightly too high in consequence of our having put $\theta = 1$.

For *Oxygen* SIEGEL [2]) has found $Q_0 = 160000$ Gr.cal. Hence $^4/_{27} Q_0 : T_k$ here becomes $= {}^4/_{27} \times (160000 : 154.25) = {}^4/_{27} \times 1040 =$ $= 154$. With $\sqrt{a'_k} = 2{,}6$ we find further:

$$\sqrt{A_k} = 2{,}6 \times \sqrt{155} = 32{,}4 = \mathbf{32},$$

which is again in excellent agreement.

In the second place we shall examine the *Halogens*.

First of all *Chlorine*. PIER [3]) found the value 113000 Gr.cal. for

[1]) Journal de Chem. Phys. **12**, 119 (1914) and **13**, 219 and 465 (1915).

[2]) Zeitschr. f. physik. Chem. **87**, 642 (1914). Cf. also BRINER, l.c.

[3]) Ibid. **62**, 385 (1908).

Q_0. According to PELLATON the critical temperature is 144° C = 417°,1 abs., so that $^4/_{27}\ Q_0 : T_k$ becomes $= {}^4/_{27} \times 271 = 40{,}2$. Hence with $\sqrt{a'_k} = 5{,}6$ we shall get about (from the critical data of DEWAR would namely follow 5,43, from those of PELLATON 5,75)

$$\sqrt{A_k} = 5{,}6 \times \sqrt{41{,}2} = 35{,}0 = \mathbf{35}.$$

Though this seems somewhat too high to us, as from 32 to 30 may be expected, the order of magnitude is yet again in agreement with what was found for it for $\sqrt{A}$ with other elements. Possibly Q_0 is somewhat too high, or also $\sqrt{a'_k} < 5{,}6$; and perhaps in this case — where $a_k : a_1$ will be further from unity than for H_2, N_2 and O_2, which have so much lower critical temperatures — θ will also be so much smaller than 1, that the found value 35 will have to be lowered to $\pm$ 32.

With regard to *Bromine* PERMAN and ATKINSON [1]) found for Q_0 the value 57000 Gr.cal. With $T_k = 302°{,}2$ C = 575,3 abs. we have $^4/_{27} Q_0 : T_k = {}^4/_{27} \times 99 = 14{,}7$. We calculate with $\sqrt{a'_k} = 7$:

$$\sqrt{A_k} = 7 \times \sqrt{15{,}7} = 27{,}7 = \mathbf{28},$$

a value that is very plausible.

In conclusion *Iodine*. STARK and BODENSTEIN (1910) found for it $Q_0 = 35500$ Gr.cal., so that with $T_k = 512°$ C = 785°,1 abs. we find $^4/_{27} \times 45{,}2 = 6{,}7$ for $^4/_{27}\ Q_0 : T_k$. With $\sqrt{a'_k} = 9$ this gives: accordingly:

$$\sqrt{A_k} = 9 \times \sqrt{7{,}7} = 25{,}0 = \mathbf{25}.$$

Nor is this value, though somewhat small, at all impossible.

6. Conclusion.

It has, indeed, appeared very clearly from the above, that **the heats of dissociation Q_0, on the decomposition of the molecules H_2, N_2, O_2, Cl_2, etc. into their atoms, are perfectly accounted for by the increased valence attractions $\sqrt{A}$ of the separate atoms found by us in earlier papers.**

By means of (1) or (1^a) we are henceforth able to calculate Q_0, when $\sqrt{a}$ and $\sqrt{A}$ are known or reversely to compute $\sqrt{A}$ according to (2), when Q_0 is known.

In GIBBS-PLANCK's well-known formula for the dissociation of gases [2])

---

[1]) Ibid. **33**, 215 (1900).

[2]) Of course this formula has *nothing* to do with the so-called theorem of heat of NERNST, as many pupils and followers of this scientist erroneously think (cf. many articles in many periodics). The formula was already given in nuce by GIBBS in 1878, and was later frequently elaborated by PLANCK (1887), v. D. WAALS (1891), myself (1892), DUHEM (1893) and many others for different cases. [Cf.

$$\log K = -\frac{Q_0}{RT} + \Sigma\left[\nu_1\left(\frac{c_1}{R} + 1\right)\right]\log T + (\Sigma\nu_1)\log p + \Sigma(\nu_1 C_1), \quad (3)$$

in which K represents the constant of dissociation in the relation $\frac{k_2^{\nu_2}}{k_1^{\nu_1}} = K$ (for di-atomic gases, where $k_2 = \frac{2x}{1+x}$, $k_1 = \frac{1-x}{1+x}$, $\frac{4x^2}{1-x^2}$ will be therefore $= K$, when x represents the so-called degree of dissociation and k_1 etc. the so-called molecular concentrations), we cannot only theoretically get nearer to the constants C_1 etc. (the so-called *chemical constants* of the components), and so also to $\Sigma(\nu_1 C_1) = C$ — this has been of late done by LORENTZ, PLANCK, SACKUR, TETRODE and others in virtue of considerations of probability in connection with the so-called theorem of heat of NERNST and PLANCK's theory of quanta — but we can also calculate the heats of reaction Q_0 for $T = 0$. Up to now we had to be satisfied with determining Q_0, just as $C = \Sigma(\nu_1 C_1)$, experimentally from a few values of x, but now we should be able to calculate the value of $\log K$ at given temperature and pressure accurately for every gas reaction, as soon as only the values of the chemical constants and of $\sqrt{A}$ and $\sqrt{a}$ are accurately known for *every element separately*.

This must henceforth be the task of physicists: to get to know *these* values completely. They are essential for the knowledge of the behaviour of the chemical substances reacting on each other. When we are further acquainted with all the values of b for the different elements, then T_k and p_k are known of every simple or compound substance, hence also their further thermical behaviour.

---

among others my *Lehrbuch der math. Chemie* (BARTH, 1901), p. 1—13, 25—28, and the "*Sechs Vorträge*" (VIEWEG, 1906), p. 64 et seq. These latter appeared originally in the Chem. Weekbl. 1905].

NERNST has only said something about the constant $\Sigma(\nu_1 C_1) = C$ — which is in connection with the constants of entropy — *in reference with his theorem*. This enabled him namely to bring the said quantities C_1 (the "chemical constants") in *connection* with the constants of the *equations of the vapour pressure* at very low temperature. But all this has of course nothing to do with formula (3) itself, which is *quite independent* of the theorem of heat. The latter says only something concerning the approach to 0 of entropy, specific heat etc. in condensed systems, in connection also with PLANCK's theory of quanta.

When formula (3) (p constant) is differentiated with respect to T, we get:

$$\left(\frac{d\log K}{dT}\right)_p = \frac{Q_0}{RT^2} + \frac{\Sigma\{\nu_1(c_1/R+1)\}}{T}.$$

And as $\left(\frac{d\log K}{dT}\right)_p = \frac{Q}{RT^2}$, $Q_{p=const.} = Q_0 + \Sigma[\nu_1(c_1 + R)]T$, hence $Q_{v=const.} = Q_0 + \Sigma(\nu_1 c_1)T$, in perfect agreement with (c) of § 4.

As regards the values of b, and of $1/a$ and $1/A$, I have already started an examination of them, and I hope I shall be able to continue this work.

Whether after all a substance as H_2 or Na at ordinary temperatures and pressures will occur in the form of *molecules*, e.g. H_2, or in that of *free atoms*, as Na, depends entirely on the values of C and Q_0 (hence on $1/a$ and $1/A$). The greater Q_0 will be and the smaller C, the smaller will be the value of K, hence of x — the sooner therefore the substance under consideration will occur in the state of molecules, and *not* in that of atoms.

I will still point out that the coefficient of *log T*, viz. $\Sigma\left[\nu_1\left(\frac{c_1}{R}+1\right)\right]$, is erroneously stereotypically put $=1{,}75$ by NERNST, POLLITZER and many pupils of NERNST. For $2HI \rightarrow H_2 + I_2$ this coefficient will be $=-2\times{}^7/_2+1\times{}^7/_2+1\times{}^7/_2=0$ (as is, indeed, sufficiently known), hence not $=1{,}75$. For $H_2 \rightarrow 2H$ it will be $=-1\times{}^7/_2+2\times{}^5/_2=1{,}5$, which again is not $=1{,}75$! The same thing applies to many other reactions. It will, therefore, be advisable to determine the value of the coefficient under consideration separately for every reaction. The same thing applies to the constant C. It will not do to assume the chemical constants C_1 all in the neighbourhood of 3; these will, indeed, also no doubt be different according to whether we have to do with a mon-atomic or a di-atomic substance, which theory indeed confirms.

Many values of Q_0 are inaccurate because they have been calculated from experiments by means of a formula with faulty coefficients (1,75 and wrong values of C); it will, therefore, deserve recommendation to calculate for a reaction not only the values of Q_0, but also those of the coefficient in question and of the constant $C=\Sigma(\nu_1 C_1)$ from the *experiments themselves*. Only in this way is it possible to obtain accurate experimental values of Q_0.

Since in the computations of § 5 we could not always reckon with perfectly reliable values of Q_0, the values found for $1/A$ are of course not *perfectly* accurate. Also in connection with our putting $\theta=1$, which will also not be perfectly true, the agreement between the found values of $1/A$, and those which we could expect (see table II) in virtue of the values found already (in an entirely independent way) may be considered very remarkable indeed.

La Tour près de Vevey. August 1918.

Physics. — "*On the measurement of very low temperatures.* XXIX. *Vapour-pressures of oxygen and nitrogen for obtaining fixed points on the temperature-scale below* 0° *C.*" By P. G. CATH. Communication N°. 152*d* from the Physical Laboratory at Leiden. (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated in the meeting of Oct. 26, 1918). [1])

§ 1. *Introduction.* The measurements of Comm. N$^{os.}$ 152*a* and *b* (1917) conducted in conjunction with Prof. KAMERLINGH ONNES (these Proc. XX pp. 991, 1155, 1160) and those of Comm. N°. 152*c* carried out with Prof. KAMERLINGH ONNES and J. M. BURGERS (these Proc. XX p. 1663) offered the welcome opportunity of making some determinations of vapour-pressures of oxygen and nitrogen using the apparatus and measuring instruments which had been fitted up and used for obtaining and measuring constant temperatures. KAMERLINGH ONNES and BRAAK (1908) (Comm. N°. 107*a*, these Proc. XI (1) p. 333) had added to the available fixed points on the temperature-scale the boiling point of oxygen and a couple of lower points, determined under sufficient guarantees as to accuracy. The determinations mentioned above were intended to supplement the latter and might serve to obtain an independent calibration of the oxygen-[2]) and nitrogen vapour-pressure thermometers, founded directly [3]) on the temperature-scale of KAMERLINGH ONNES and his collaborators.

Prof. KAMERLINGH ONNES was good enough to invite me to avail

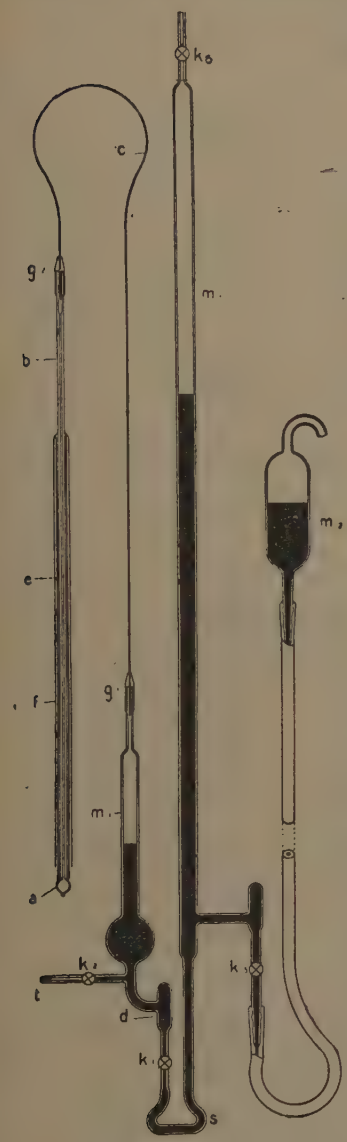
---

[1]) This paper was originally presented in the meeting of June 1917; in being prepared for the press it underwent some slight modifications on points of secondary importance.

[2]) The oxygen vapour-pressure thermometer is often used in the form given to it by STOCK and NIELSEN (Ber. d. D. Chem. Ges. 2 (1906) p. 2066) and is then usually simply called STOCK's thermometer. For accurate measurements it is necessary to resort to more suitable forms of the instrument, such as the one described in this paper. The scale given by STOCK and NIELSEN is in need of very considerable corrections (Comp. G. HOLST, Comm. N°. 148*a*; these Proc. XVIII (1) p. 829).

[3]) VON SIEMENS (vid. G. HOLST Comm. N°. 148*a*) determined the temperatures at which he observed the vapour-pressures by means of a platinum-thermometer, whose readings were reduced to those of $Pt_{I'}$ of KAMERLINGH ONNES and his collaborators, chiefly by means of four calibration-points in such a manner that the scales were made to coincide at the boiling point of oxygen as determined by KAMERLINGH ONNES and BRAAK.

myself of this opportunity. I am happy to express my sincere thanks to him on this occasion for his assistance in the investigation.[1])



§ 2. *Details of the determinations.* The temperatures were measured with the helium thermometer described in Comm. N°.152*a* and were reduced in the same way as on that occasion to a scale which approaches the absolute scale as closely as possible (Avogadro-scale of helium according to a future communication by CATH and KAMERLINGH ONNES).

The vapour-pressure apparatus agreed in its main features with model *A* used by KAMERLINGH ONNES and BRAAK (Comm. N°. 107*a*), in particular I adhered to the use of the copper tube the purpose of which is to prevent a lower temperature existing anywhere in the apparatus but in the bulb where the liquefied gas collects, at which spot the temperature of the bath is measured. As in the previous experiments mentioned use was made of the advantage offered by the apparatus of allowing different quantities of gas to be condensed in the bulb. As regards the manometer the modification was adopted which HOLST introduced in his vapour-pressure measurements of methyl chloride (Comm. N°. 144*b* Sept. 1913): the moveable tube connected to the fixed tube by a rubber tube is replaced (see fig. 1[2]) by a fixed manometric tube[3]) to which a mercury vessel is attached by means of a rubber tube. This fixed tube can be exhausted or it

[1]) Part of the present paper is embodied in P. G. CATH, Dissertation Leiden 1917, which appeared a few months after the presentation of this communication (see also note 1).

[2]) The letters are for the greater part the same as in fig. *A* Comm. N°. 107*a*, to which we may refer for the further description; a more complete explanation than that given above seems unnecessary.

[3]) The protection from temperature-changes was obtained by packing in wool. The temperature of the mercury column was determined by means of thermometers suspended beside the tube at the upper and lower ends of the mercury column.

can be connected to a space of constant pressure and a barometer through the stopcock k_6. The ground joint t may be used for exhaustion, for drying with the aid of a moisture-catcher placed in liquid air and for filling.

The oxygen was prepared from potassium-permanganate in an apparatus which could be evacuated by means of a Toepler-pump and dried by freezing with liquid air. The gas was condensed in a small bulb which was attached to the apparatus and cooled in liquid air, whence it was conveyed to the vapour-pressure apparatus by means of the air-pump [1]).

The nitrogen was prepared by Dr. LEMKES under supervision of Prof. VAN ITALLIE, to whom I express my sincere thanks.

§ 3. *Results.* The results for oxygen were as follows:

TABLE I. Vapour-pressures of oxygen.

	θ	T	p in int. c.m. [2])	p in int. atm.	p calculated	ΔT O—C
I	— 182°.62 C.	90°.47 K.	78.663	1.0350	1.0351	0.00 deg.
II	182.88	90.21	76.560	1.0074	1.0093	+ 0.02
III	183.22	89.87	73.867	0.9719	0.9725	+ 0.01
IV	183.91	89.18	68.776	0.9050	0.9053	0.00
V	186.91	86.18	49.330	0.6491	0.6491	0.00
VI	192.01	81.08	21.319	0.3463	0.3463	0.00
VII	195.50	77.59	16.215	0.2138	0.2137	0.00
VIII	201.38	71.71	6.401	0.08423	0.08398	— 0.02
IX	204.52	68.57	3.611	0.04752	0.04753	0.00
X	210.72	62.37	0.959	0.01262	0.01286	+ 0.08

[1]) One third of the distillate was used for rinsing out the apparatus, the second third for the filling.

[2]) In order to prevent possible misunderstanding it is mentioned here that the pressure is given in "international centimeters" i.e. in the 76th part of an international atmosphere or in other words as the height of a mercury-column measured not at Leiden itself (local cm.), but at a place where one international atmosphere exactly agrees with 76 cms mercury at 0° C., i.e. where $g = g_{\text{Bur. Int.}} : 1.0003322$, the value which is taken as $g_{\text{norm.}}$ and at present is to be put at 980.615. (At Leiden $g = 981.276$ and 1 int. atm. $= 75.9488$ cms). For the definition of international atmosphere Comp. Leiden Comm. Suppl. 23. Mathem. Enc. V 10. Einh. *a*. When in these Communications about observations at Leiden an atmosphere pure and simple is mentioned, the international atmosphere is always meant.

The last two columns show, that the observations are very well represented by an equation of the form

$$lg\, p = \frac{A}{T} + B + CT \quad (p \text{ in int. atm.})$$

with $A = -419.31$, $B = 5.2365$, $C = -0.00648$, kindly calculated for me by Professor VERSCHAFFELT. According to the equation the normal boiling point of oxygen would be

$$T = 90°.13 \text{ K.}, \qquad \theta = -182°.96 \text{ C.},$$

whereas the observations in the immediate neighbourhood of the boiling point themselves give:

$$T = 90°.14 \text{ K.}, \qquad \theta = -182°.95 \text{ C.}$$

This boiling point was found to be $T = 90°.11$ K. [1]) by

TABLE II. Vapour-pressures of oxygen.
Comparison with KAMERLINGH ONNES and BRAAK and with v. SIEMENS (corr.).

T	p (in atm.) (K. O. and B.)	p (in atm.) (v. S. corr.)	p (in atm.) (CATH)	$\triangle T$ (K.O., B.—C.)	$\triangle T$ (v. S.—C.)
90°.69	1.0618		1.0583	—0.035 deg.	
90.47		1.036	1.0351		—0.01 deg.
90.21		1.010	1.0093		0.00
90.11	1.0002		0.9975	—0.03	
89.87		0.9742	0.9725		—0.01
89.18		0.9059	0.9053		—0.01
86.56	0.6792		0.6777	—0.02	
86.18		0.6500	0.6491		—0.01
83.66	0.4819		0.4808	—0.02	
81.08		0.3463	0.3463		0.00
77.59		0.2137	0.2137		0.00
71.71		0.08390	0.08398		0.00
68.57		0.04695	0.04753		+0.05
62.37		0.01272	0.01286		+0.04

[1]) Comm. N°. 107*a* gives $T - T_{0°C} = -182°.986$C. In the computation, however, the pressure-coefficient of hydrogen at 1100 mm. freezing point pressure is taken at 0.0036627, whereas, if we put $T_{0°C} = 273°.09$ K. and adopt the corrections to the absolute scale given by KAMERLINGH ONNES and BRAAK, the correct value is 0.0036628, which leads to —182.98 and $T = 90°.11$ K.

KAMERLINGH ONNES and BRAAK and $T = 90°.12$ K by HENNING.[1]).

In Table II the results of KAMERLINGH ONNES and BRAAK and those of v. SIEMENS, the latter after the reduction of his temperature-measurements to those of KAMERLINGH ONNES and his collaborators has been corrected according to HOLST[2]), are compared with the formula which represents my observations[3]).

The results show that except for the lowest points HOLST's reduction of v. SIEMENS' observations has been very successful.

The results for *nitrogen* are given in Table III.

TABLE III. Vapour-pressures of nitrogen.

N°.	θ	T	p (in int. cm.)	p (in atm.)	p (calc.)	ΔT ($O-C$)
I	— 188°.88 C.	84°.21 K.	159.11	2.0942	2.0948	0.00 deg.
II	190.66	82.43	133.44	1.7558	1.7539	0.00
III	193.91	79.18	93.86	1.2350	1.2382	0.02
IV	195.84	77.25	75.19	0.9893	0.9904	0.01
V	198.26	74.83	56.13	0.7385	0.7366	— 0.02
VI	201.55	71.54	36.16	0.4758	0.4756	0.00
VII	204.69	68.40	22.837	0.30049	0.30031	0.00
VIII	208.58	64.51	12.090	0.15908	0.15882	— 0.01
IX	213.14	59.95	4.695	0.06178		
X	215.20	57.89	2.1882	0.03792		

[1]) Ann. d. Phys. (4) 40 (1913) p. 635. The theoretical correction to the readings on the scale of the hydrogen-thermometer given by BERTHELOT which HENNING applies, does not differ appreciably in this region of low temperatures from the experimental correction found by KAMERLINGH ONNES and BRAAK. All the same it is doubtful whether HENNING's value may be looked upon as a final confirmation of the boiling point found by KAMERLINGH ONNES and BRAAK, since his measurement contains an uncertainty to which attention was drawn in note 1 to p. 998 of these Proc. XX (2) (Comm. N°. 152*a*).

[2]) In this correction HOLST puts the temperatures at the higher points 0.02 and at the lowest point 0.01 of a degree higher than KAMERLINGH ONNES and BRAAK, which is due to the fact that in the reduction he uses his own formula of interpolation instead of the observations themselves. This explains the difference of the deviations at the top of the two columns for coinciding points, although v. SIEMENS in his determinations takes those of KAMERLINGH ONNES and BRAAK as his starting point.

[3]) As observed in Comm. N°. 107*a*, there is a systematic difference of 0.10 degree at the boiling point and 0.13 degree at the lower points between the observations

The observations I to VIII refer to liquid nitrogen, the last two to solid nitrogen. The last two columns show that for liquid nitrogen the relation between vapour-pressure and temperature can be successfully represented by an equation of the same form as for oxygen, with

$A = -334.64$, $B = 4.6969$, $C = -0.00476$ (p in int. atm.);

according to this equation the normal boiling point of nitrogen is

$$T = 77°.31 \text{ K.} \qquad \theta = -195°.78 \text{ C.}$$

FISCHER and ALT [1]) give -195.67 on the hydrogen scale, on the basis of $\alpha = 0.0036625$. With $\alpha = 0.0036627$ the result changes to $-195°.66$, whence $\theta = -195°.61$ C. With the smaller dimensions of their thermometer a smaller accuracy was to be expected. They give their accuracy as being $\pm .05$ of a degree [2]).

TABLE IV. Vapour-pressures of nitrogen.
Comparison with v. SIEMENS (corr.) and with HAMBURGER and HOLST (corr.).

T	p in atm. (v. S. corr.)	p in atm. (H. en H.)	p in atm. (CATH)	$\triangle T$ (v. S.—C.)	$\triangle T$ (H.H.—C.)
84°.21 K.	2.099		2.0948	—0.02 deg.	
82.43	1.756		1.7539	0.00	
80.93*		1.4983	1.4894		—0.06 deg.
79.18	1.236		1.2382	+0.01	
78.55*	1.1561	1.1561	1.1460		—0.03
77.25	0.9895		0.9904	0.01	
74.83	0.7331		0.7366	0.04	
72.15*		0.5222	0.5143		—0.06
71.54	0.4706		0.4756	0.09	
69.34*		0.3484	0.3439		—0.02[5]
68.40	0.2952		0.30031	0.10	
64.51	0.1546		0.15882	0.14	
59.95	0.05958		0.06178	0.17	
57.89	0.03618		0.03792	0.19	

by TRAVERS, SENTER and JAQUEROD (London Phil. Trans. A. 200 (1902)) and those of KAMERLINGH ONNES and BRAAK, the latter being lower. The former are therefore now only of importance historically.

[1]) Ann. d. Phys. (4) 9 (1902) p. 1149.

[2]) The uncertainty in their determination seems to have been larger, for as

The two measurements for solid nitrogen give the relation

$$\log p = -\frac{358.73}{T} + 4.7769;$$

as holding near the triple-point; this combined with the equation for liquid nitrogen gives for the triple point (i.e. the point of intersection of the two vapour-pressure curves).

$T = 63°.23$ K, $p = 0.1269$ int. atm. $= 9.64$ int. cm.

whereas v. SIEMENS, account being taken of HOLST's corrections, found

$T = 63°.25$ K. $p = 9.35$ cm.

In Table IV v. SIEMENS' results — after correction according to HOLST of the reduction to the scale of KAMERLINGH ONNES c.s. — and those of HAMBURGER and HOLST to which I have applied a small correction [1]) are compared with the equation which represents my observations.

Finally I have tried to connect my results to CROMMELIN's measurements between the boiling point and the critical point [2]). The latter may be represented satisfactorily [3]) by the equation [4])

---

regards my measurements it seems improbable that the uncertainty in the determination of the temperature, independently of systematical errors, has been larger than about 0.01 of a degree (cf. table III), and — in view of the recent determinations with the instrument, which will be published in the next communications — the systematical error of the Leiden gasthermometer (apart from the effect of errors in the expansion of the glass and in the constant for the capillary depression of the mercury) also may be put at about 0.01 of a degree.

With regard to the differences in Table IV it should be borne in mind, that the observations of v. SIEMENS as well as those of HAMBURGER and HOLST are based on the readings of a platinum thermometer, which has not been compared directly with the gasthermometer. They cannot serve therefore to estimate the systematical error in the temperature-determination. [This note has been somewhat modified in the translation].

[1]) HAMBURGER and HOLST (these Proc. XVIII 1 (1915) p 872) obtained their results by reducing the readings of their platinum-thermometer to the vapour-pressures as found by KAMERLINGH ONNES and BRAAK (comp. note 2 page 660). As the authors also give the temperatures at the oxygen-pressures observed by them and as at 78°.42 K. their temperature is 0.05 degrees higher than my results for oxygen which are at present the only direct measurements of sufficient accuracy in that region, I have applied a correction of that amount to their temperatures. These are marked with an asterisk.

[2]) Comm. N°. 145*d*, these Proc. XVII (2) p. 959.

[3]) We leave out of account the reading at 81°.21 K., which CROMMELIN himself marked as inaccurate by placing it in square brackets.

[4]) The agreement with this formula containing two terms of the expansion of $T \log p$ in powers of T is not inferior to what is obtained by means of three terms of the expansion in $1/T$ as given by CROMMELIN. The latter formula, however,

$$T \log p = A + B(T - T_k) + C(T - T_k)^2$$

where $A = 190.86$, $B = 3.9649$, $C = 0.00100$ for p in int. atm. This equation is again due to the kindness of Prof. VERSCHAFFELT.

Table V gives the comparison between observation and calculation.

For $T = 81°.87$ K. the equation gives $p = 1.6363$ (CROMMELIN), whereas $p = 1.6584$ (CATH) is found from the equation which represents my observations. The

TABLE V. Vapour-pressures of nitrogen (CROMMELIN).

T	p (in atm.)	p calc. II.	ΔT(O—C II)
125°.96 K.	33.49	32.728	−0.09 deg.
124.24	30.364	30.219	0.00
120.98	25.889	25.892	0.06
117.62	21.820	21.906	0.04
111.78	15.949	15.993	−0.01
99.51	7.3705	7.3652	0.04
93.91	4.8278	4.8467	−0.06
90.62	3.7248	3.7043	−0.02
86.21	2.5067	2.5016	

difference between these two values (corresponding to $\Delta T'_{Cr-Ca} = .10$ of a degree) is so large, that for the present it seems impossible to combine the observations above the boiling point with those below it in one and the same equation.

satisfies the condition of giving the correct critical pressure at the critical point. The present formula gives $p_k = 32.728$ atm. instead of the observed value $p_k = 33.49$.

Chemistry. — *"On Phenyl Carbaminic Acid and its Homologues"*. By Prof. F. E. C. SCHEFFER. (Communicated by Prof. BÖESEKEN.)

(Communicated in the meeting of September 29, 1918).

1. *Introduction.*

It was observed by DITTE in 1887 that under high pressure and at temperatures lower than room temperature aniline with carbonic acid can react under formation of a solid compound which consists of equal molecular quantities of aniline and carbonic acid [1]). It may besides be inferred from his paper that unmixing takes place at ordinary temperature. Some years ago Dr. J. J. POLAK carried out a number of experiments with the same system of substances in the organic chemical Laboratory of the Amsterdam University [2]); he too succeeded in ascertaining the existence of a compound, and the result of his analysis pointed to the same composition as was given by DITTE. It further appeared in his researches that the compound melts on being heated in the presence of a liquid and a vapour rich in carbonic acid, before the meniscus liquid-vapour disappears, with formation of a second liquid layer; this suggested the thought to me that the system aniline-carbonic acid would present an analogy in its behaviour with BAKHUIS ROOZEBOOM's gas-hydrate systems and with the system sulphuretted hydrogen-water, the phenomena of which I have fully described in These Proceedings [3]). It will appear from the below-mentioned observations that the compound, which in my opinion is to be considered as a carbaminic acid, gives rise in the P-T diagram to the appearance of a quadruple point, where solid compound, two liquid layers, and gas coexist, and that the three-phase lines which intersect in this quadruple point, can be determined with sufficient accuracy. This system also furnishes a new application of the quadruple point rule, drawn up by me in 1912 [4]), which was described by SCHREINEMAKERS in the Zeitschrift für physikalische Chemie almost at the same time [5]).

---

[1]) Compt. rend. **105**. 612. (1887).

[2]) Not published. The results of his research have been kindly put at my disposal by Dr. POLAK, for which I gladly express my indebtedness to him here.

[3]) These Proc. **13**. 829 (1910/11) and **14** 195 (1911/12).

[4]) These Proc. **15**. 389 (1912/13).

[5]) Zeitschr. f. physik. Chem. **82**. 59 (1913).

I further extended this investigation to the three toluidines, and I have succeeded in determining the limits of existence of three compounds. I have determined the composition of two of these compounds by analysis; the composition of the third had already

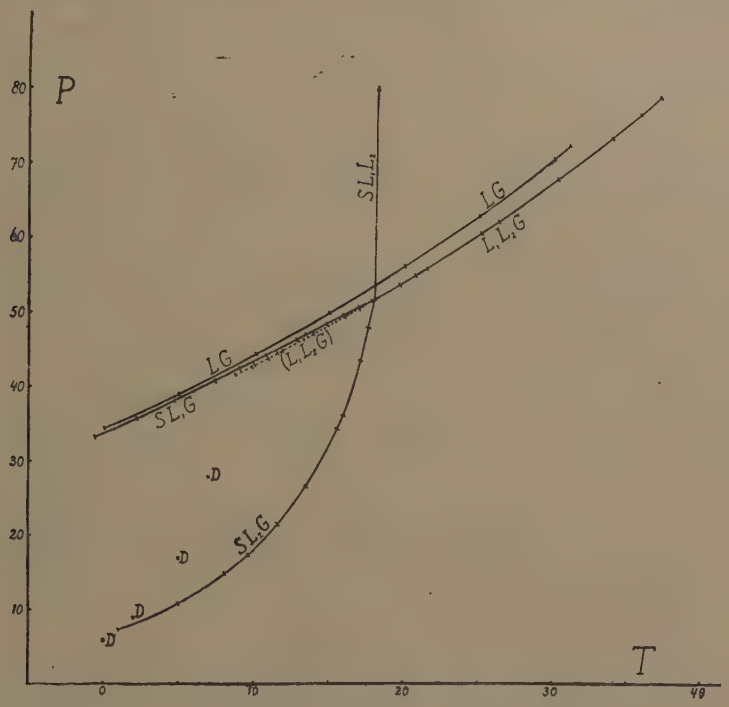


Fig. 1.

been found by DITTE. The great analogy between the three systems will probably justify the conclusion that these compounds are the three isomer tolyl carbaminic acids.

2. *The system aniline-carbonic acid; the phenyl carbaminic acid.* From "anilin purissimum" of MERCK coloured red by contact with the air a middle fraction was separated by fractionation; a slight quantity of this was put into the Cailletet tube by the aid of a long glass capillary. As a test tube I used a tube with a widened upper end of the same shape as in my researches on the systems ether-water and hexane-water [2]). For the filling with carbonic acid the test tube was connected with a ground piece to an apparatus consisting of a generating apparatus for carbonic acid, which was obtained from diluted sulphuric acid and sodium bicarbonate, and was dried with phosphorus pentoxide,

[1]) These Proc. **15**. 380 (1912/13),
[2]) These Proc. **16**. 404 (1213 14).

an apparatus for a high vacuum, consisting of a vessel with cocoa-nut carbon and a GEISSLER tube, a tube for condensation of the carbonic acid (by the aid of liquid air) and a vessel of about $^1/_2$ liter capacity as carbonic acid reservoir [1]). As it was exclusively my purpose in these experiments to determine three-phase pressures, a determination of the concentration of the mixtures used was superfluous. For the first observation I used a mixture with great excess of carbonic acid. When the test tube had been screwed on to the pressure cylinder after the filling, and the mixture had been heated to the ordinary temperature, it appeared that the solid substance could be kept at ordinary temperature only under high pressure. When the available volume was so small that there was only a small quantity of gas present, then a three-phase equilibrium of compound by the side of a thinly liquid layer (rich in carbonic acid) and gas occurred at the ordinary temperature at a pressure of about 50 atmospheres. When the pressure was diminished, the liquid vanished with violent boiling, and solid remained by the side of gas. Below 30 atmospheres the solid substance decomposed with formation of a liquid rich in aniline, a strong generation of gas being perceptible in this layer. Accordingly the solid compound is decomposed into liquid and gas on decrease of pressure. It is clear that the pressure at which this decomposition just sets in indicates the three-phase coexistence of compound, liquid rich in aniline, and gas. On increasing enlargement of the volume there remains coexistence of liquid by the side of gas. At the ordinary temperature the existence of phenyl carbaminic acid is, therefore, only possible at pressures above about 30 atmospheres. Hence in perfect analogy with the gas hydrates the dissociation tensions of this compound are three phase equilibria. This applies also to the determinations which DITTE carried out by observation of the pressure at which gas begins to form from the crystals, or of the pressure at which this generation ceases, which are of course theoretically the same, but practically different according to DITTE [2]). DITTE ascribes the latter to inaccuracies of the temperature determination, in my opinion the slowness of the transformation $S \rightarrow L + G$ is undoubtedly responsible for this. In my former researches on the system sulphuretted hydrogen-water I have also been able to observe such a slow transformation [3]).

In order to be able to determine the three-phase pressures

---

[1]) Cf. also These Proc. **13**. 830 (1910/11).

[2]) loc. cit.

[3]) loc. cit.

accurately, I have adjusted a wide cylindrical vessel narrowed at the lower end round the test tube, in which vessel alcohol was stirred by means of vertically moving leaden plates; the heating took place electrically; cooling was effected by introduction of solid carbonic acid. In the observations with small volume it now appeared that the maximum temperature at which the compound can exist by the side of gas, is about 18°. The quadruple point lies at this temperature; the pressure is about 52 atmospheres. In this point there is intersection of the three-phase lines S (compound), L_1 (liquid rich in carbonic acid), and G (gas), which is stable at temperatures below the quadruple point, and cannot be prolonged above the quadruple point, $S + L_1 + G$, which exhibits a value of $\frac{dp}{dT}$ which rapidly increases with the temperature in the neighbourhood of the quadruple point, $L_1 + L_2 + G$, which indicates stable equilibria above the quadruple point, but can also be easily determined below the quadruple point; then these equilibria are, however, metastable with respect to the solid phase. The fourth three-phase line $S + L_1 + L_2$ rapidly moves from the quadruple point to higher pressure. The situation of the three-phase lines is indicated in fig. 1 by the letters given above; the quantitative data have been collected in table 1; they have been obtained with two mixtures; one contained a great, the second a small excess of carbonic acid; the determinations carried out with the two mixtures, are in good agreement. When the figure is consulted the phenomena described in the beginning of this paragraph will be clear. As long as gas is present, the solid compound can only exist for pressures which are higher than the three-phase line SL_2G. DITTE's determinations, which are indicated both in the figure and in the table by the symbol D, appear to depart perceptibly from mine; only in the neighbourhood of 0° do the observations agree fairly well. It makes the impression that DITTE has determined the points where solid substance is formed on increase of pressure, and that the pressures have been found much too high through the retardation of the transformation $L + G \rightarrow S$, though DITTE mentions that he observed the pressures at which the generation of gas ceased. In this respect the phenomena are again in perfect analogy with the system sulphuretted hydrogen-water, where CAILLETET and BORDET's observations present analogous deviations with mine. [1]) DITTE does not lay claim, however, to great accuracy for his observations; he states that his determinations give

[1]) These Proc. **13**. 833, fig. 2 and table on p. 834. (1910/11).

only rough values, but that they may yet give an impression of the way in which "aniline carbonate" dissociates.

TABLE 1.

$LG(CO_2)$		L_1L_2G		SL_2G	
T	P	T	P	T	P
0	34.4	8.7	41.8	0	6 D
5.0	39.0	9.8⁵	42.8	0.9⁵	7.4
10.1	44.6	10.0	43.2	2	9 D
15.0	50.2	10.9	43.9	5.0	10.9
20.0	56.6	11.5	44.8	5	17 D
25.0	63.4	11.8⁵	45.1	7	28 D
30.0	71.1	13.2⁵	46.6	8.1	15.0
31.1	72.9	13.9⁵	47.4	9.7	17.5
SL_1G		16.0	49.7	11.6	21.7
		16.9⁵	50.8	13.5	26.7
T	P	17.8	51.8	15.5	34.6
		19.7	54.1	15.9	36.4
−0.6	33.3	20.7	55.4	17.1	43.8
+2.2	35.7	21.5	56.3	17.6	48.3
4.7	38.1	25.1⁵	61.1		
7.4	40.8	26.3	62.7	SL_1L_2	
9.0	42.5	30.2⁵	68.4		
10.8	44.4	33.9	74.0	T	P
12.8	46.5	35.8	77.1	18.0	52
13.4	47.3	37.1⁵	79.5	18.2	80
14.8	48.6				
15.9	49.8				
16.9⁵	50.9	Quadruple point		18.0°	52.0 atm.

When we pursue the three-phase line L_1L_2G towards higher temperature, the fluidity of the upper layer becomes greater and greater, and at 37° the critical phenomenon presents itself; the critical end-point lies 6° and about 7 atmospheres higher than the critical point of carbonic acid. We further derive from the figure that the three-

phase tension L_1L_2G begins to depart more from the carbonic acid tension with increasing temperature; at the quadruple point the deviation amounts almost to 2 atmospheres; at the critical point of carbonic acid to about $3^1/_2$ atmospheres.

It follows from the already mentioned quadruple rule, which I formulated before as follows: *The region that does not possess metastable prolongations of threephase lines in the P.-T.-projection, is that of coexistences of phases of consecutive concentration* [1]), that the region between SL_1G and SL_1L_2 satisfies the above mentioned condition. In this region, which besides by the two mentioned three-phase equilibria is also bounded by SL_1G and L_1L_2G resp. by SL_1L_2 and SL_2G, the coexistences occur of the two phases which the adjoining three-phase equilibria have in common, hence in this case $S + L_1$, $L_1 + G$, and $S + L_2$. As these coexistences according to the rule mentioned must refer to phases which succeed each other in concentration, the succession is GL_1SL_2; the concentration of the compound lies, therefore, between that of the two coexisting liquid layers. Hence the transformation $S \rightleftarrows L_1 + L_2$ takes place on the three-phase line SL_1L_2 in the neighbourhood of the quadruple point.

3. In order to get acquainted with the concentration of the compound Dr. POLAK has caused a weighed quantity of aniline to act on an excess of carbonic acid in a fused-to tube at the ordinary temperature. After the compound had been formed the tube was opened again at $-80°$, and placed in a bath of about $-60°$; after half an hour the tube was again fused to, and weighed after having been heated to the ordinary temperature. This analysis yielded the concentration $C_6H_5NH_2 . 1 . 01\ CO_2$.

I have carried out three analyses in a way that differs but little from that described here; the method of investigation was the same as that which I have described in my second paper on the system sulphuretted hydrogen-water [2]); the excess of carbonic acid was sucked off at $-80°$ by means of a waterjet pump. For the quantity of carbonic acid in gramme-molecules which combines with one mol. of aniline, was found successively 0.98, 0.99, and 0.98. The compound consists, therefore, of an equal number of molecules of aniline and carbonic acid.

4. *The system o-toluidin-carbonic acid; the o-tolyl-carbaminic acid.* Also in the system o-toluidine-carbonic acid I have been able to ascertain the formation of a compound; the quadruple point lies here,

[1]) loc. cit.
[2]) loc. cit.

however, at lower temperature. As the inquiry into the equilibria

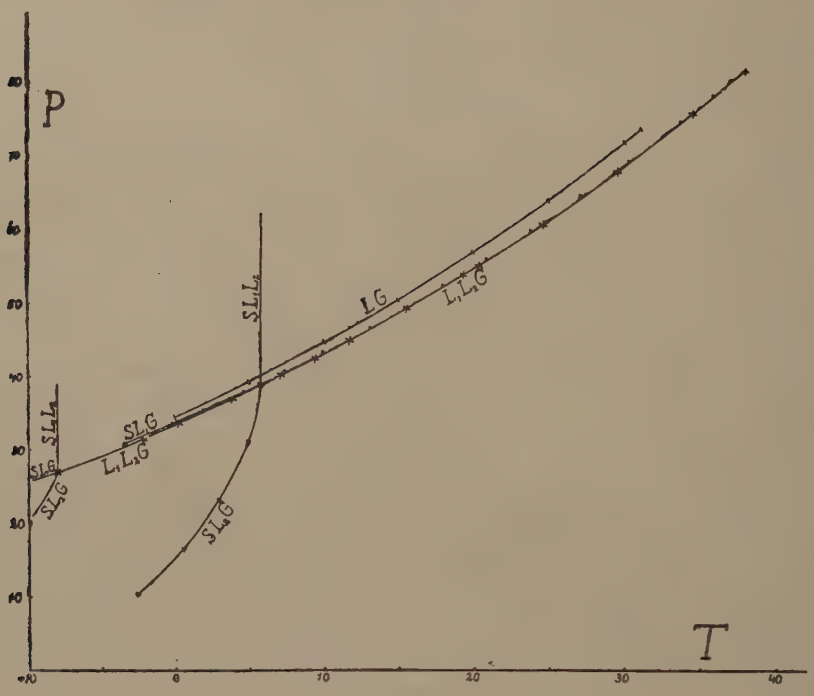


Fig. 2.

at low temperatures is attended with experimental difficulties, I have rested satisfied with the determination of the quadruple point and of the three-phase line L_1L_2G with the critical end point. The found three-phase pressures are recorded in table 2 and indicated in fig. 2

TABLE 2.

L_1L_2G

T	P	T	P
−2.1	31.5	19.4	53.5
+0.3	33.6	20.4	54.8
3.9	36.8	24.7	60.1
7.3	40.1	29.6	67.1
9.5	42.3	34.5	74.9
11.8	44.7	38.1	80.8
15.6	49.0		
Quadruple point		−7.5°	27.5 atm.

by crosses. At temperatures below the quadruple point a compound again occurs, which has also already been observed by DITTE, and which according to him consists of equal molecular quantities of o-toluidine and carbonic acid. I myself have not determined the concentration of this compound; the application of the above described method of analysis is accompanied with pretty great difficulties at the low temperatures. The possibility that the solid substance should be pure o-toluidine is excluded, because the quadruple point lies at higher temperature than the melting-point of the pure substance. (Cf. table 5).

5. *The system m-toluidine-carbonic acid, the m-tolyl carbaminic-acid.* There occurs a quadruple point SL_1L_2G in the system m-toluidine-carbonic acid at a temperature which lies between that of o-toluidine-carbonic acid and that of aniline-carbonic acid. The three-phase lines have again the same relative situation as was described above. The observations referring to the three-phase lines L_1L_2G, SL_1G and SL_2G are recorded in table 3 and indicated in figure 2 by triangles. In this figure are also found the vapour tensions of pure carbonic acid (see table 1).

6. The analysis of the compound according to the method

TABLE 3.

L_1L_2G		SL_2G		SL_1G	
T	P	T	P	T	P
7.5	40.5	−2.6	10.4	−3.4	30.8
10.0	43.1	+0.6	16.5	−0.2	33.4
13.2	46.4	2.9	23.1	+1.9	35.3
15.5	49.2	5.0	31.5	2.9	36.2
18.0	52.1			4.7	37.7
20.9	55.7			5.7	38.6
23.9	59.5				
27.3[5]	64.2				
30.3	68.5				
33.7	73.8	Quadruple point		6.3°	39.2 atm.
35.8[5]	77.4				
37.2	79.6				

mentioned in § 3 yielded varying values in contrast to those of aniline and of p-toluidine, as will appear in § 8. For the quantity of carbonic acid which combines with one molecule of m-toluidline, was found successively 0.76, 0.79, 0.85, 0.86, 0.88, 0.88, 0.93, 0.89, 0.86, 0.92, and 0.89 mol. These values for the carbonic acid content are most probably all too small. The cause of this deviation is in my opinion the following: Liquid carbonic acid and m-toluidine are little miscible. Accordingly the formation of the compound on cooling takes chiefly place on the boundary of the two layers. In consequence of this a partition of solid substance is continually formed, separating the two layers. Part of the toluidine can, therefore, be withdrawn from the action of carbonic acid. It is clear that after the excess of carbonic acid has been sucked off, the quantity of bound carbonic acid is found too small. In order to render the formation as complete as possible, the tubes were kept in ice for several days; in the successive determinations this period increases from 2 to 10 days. It appears, therefore, that the time has not much influence on the result of the analysis. In the last determination the tube was cooled for 7 hours with ice and salt (—15 to —20°); it also yields too low a result. The supposition that the m-toluidine should be impure, appeared erroneous, as the correct value of 65° was found for the melting-point of the acete compound. I think I am justified in concluding from the above-mentioned determinations that the compound likewise consists of equal molecular quantities of toluidine and carbonic acid.

7. *The system p-toluidine-carbonic acid; the p-tolyl carbaminic acid.* In the P-T-diagram the system p-toluidine-carbonic acid yields a three-phase line L_1L_2G, which deviates little from that of the said systems. (Cf. fig. 3). The quadruple point SL_1L_2G lies here at higher temperature; the four three-phase lines which intersect in this quadruple point, are indicated in fig. 3; the quantitative data in table 4. The relative situation of the four phases is the same here as in the preceding systems. The stable part of the three-phase line SL_2G terminates at lower temperature in a second quadruple point SS_BL_2G (S_B is solid p-toluidine). In the preceding systems the corresponding quadruple point lies at lower temperature and pressure; in this system the temperature of the two quadruple points differs little from the critical temperature of carbonic acid. The three-phase lines SL_2G and S_BL_2G are easy to determine, when we heat at constant pressure and read the temperature at which liquid is formed. Without further examination we may state about the three-phase

line SS_BL, that the slope will be steep. Of three of the three-phase lines that pass through the quadruple point SS_BL_2G the situation

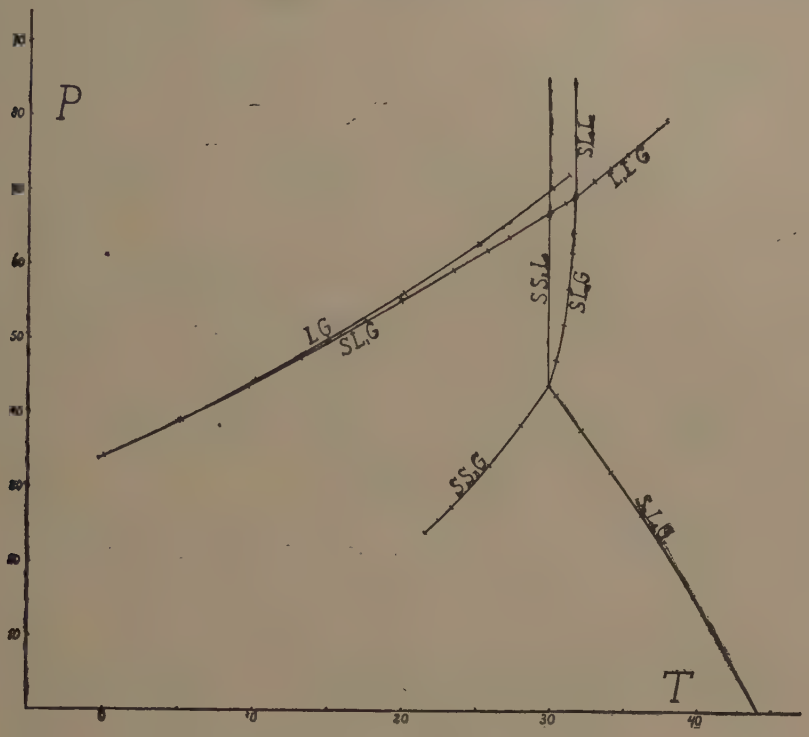


Fig. 3.

could, therefore, easily be given. The fourth three-phase line SS_BG was, however, difficult to find. The quadruple point rule mentioned, however, gave me an indication where it was to be found. There exist two possibilities for the situation of this fourth three-phase line, which are represented by fig. 4*a* and *b*. The three-phase line

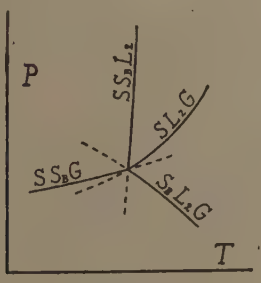


Fig. 4*a*.

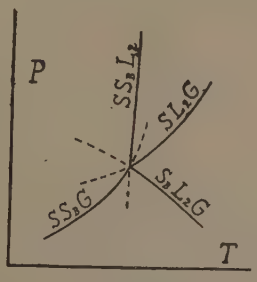


Fig. 4*b*.

SS_BG must namely lie between the metastable prolongations of SL_2G and S_BL_2G (fig. 4*a*) or between those of SL_2G and SS_BL_2 (fig. 4*b*). Other situations are impossible, because else two-phase coexistences

would occur with an angle larger than 180°; I have set forth in the mentioned paper that this is impossible. When with the aid of the quadruple point rule we examine what succession of the phases would appear according to fig. 4*a*, it appears that no metastable prolongations occur between SS_BL_2 and SL_2G, that the two-phase coexistences in this region are: $G + S$, $S + L_2$ and $L_2 + S_B$ and that the succession of the phases is given by GSL_2S_B.

In an analogous way it would follow from fig. 4*b* that the order of the phases would be SGL_2S_B. This order indicates diminishing carbonic acid content, because S_B represents solid p-toluidine. That the compound S would be richer in carbonic acid than the gas phase, which practically consists of pure carbonic acid, is excluded; p-toluidine has a very slight vapour tension (b.pt. 200°) at this temperature, and the content of p-toluidine in the vapour is, therefore, very small. The only possibility is, therefore, given by fig. 4*a*. I have, therefore,

TABLE 4.

SL_1G		L_1L_2G		SL_2G	
T	P	T	P	T	P
—0.4	33.9	32.8	72.0	30.2	47.7
+5.1	39.1	33.9	73.9	30.8	52.5
9.6	43.6	35.0[5]	75.7	31.1	57.3
13.1	47.6	37.7[5]	80.4	31.3	62.2
17.5	52.7			31.4	65.0
19.8	55.5	SS_BG		S_BL_2G	
23.3	59.8	T	P		
25.6	62.4			T	P
27.0	64.3	21.5	24.2		
29.7	67.7	23.3	27.6	30.3	43.0
30.8	69.0	25.8	33.4	32.0	38.2
		27.9	38.8	34.0	32.5
				36.2	26.8

Quadruple point			
	SL_1L_2G	31.5°	70 atm.
	SS_BL_2G	29.7°	44 atm.

tried to find the required three-phase equilibrium in the region between

the metastable prolongations of SL_2G and S_BL_2G. The determinations were difficult; the only way to find the equilibria was by examining whether rise of pressure or descent of pressure takes place at constant temperature after some lapse of time. It is clear that below the three-phase line SS_BG (fig. 4*a*) the two-phase coexistence $S_B + G$ is found, and above it $S + G$ and $S + S_B$, because S with regard to its concentration lies between G and S_B, and the coexistence of the two solid substances will extend towards higher pressure [1]). Hence the transformation $S_B + G \rightleftarrows S$ occurs on the three-phase line. The upper arrow indicates the conversion on diminution of volume, the lower one on expansion. When at a definite temperature a fall of the pressure takes place, we are above $SS_B G$; when the pressure increases, we are below $SS_B G$. In this way an upper and a lower limit was found, which were no further apart than one atmosphere, sometimes some tenths of an atmosphere. The slowness of the transformation rendered this method of working necessary; the lower limit was found to yield values which were better reproducible than the upper one. The explanation of this is in my opinion to be found in the fact that the transformation $S \rightarrow S_B + G$ takes place more easily than the opposite one. This is self-evident, as the action of G on S_B can exclusively take place on the boundary of the two phases, and formation of a phase S can stop the action. Accordingly the values of the lower limit are recorded in table 4; besides, the upper limit often differs no more than a fraction of an atmosphere from the lower one, as has been said. The relative situation of the three-phase lines is actually that which was predicted with the aid of the quadruple point rule.

8. The analysis of the compound did not present any difficulties. The results of the analyses were resp. 1.00, 0.97, and 0.99 mol. CO_2 to 1 mol. p-toluidine. Hence the compound contains equal molecular quantities of the two components.

9. *Summary of the results.*

The four examined systems yield pretty well coinciding three-phase lines L_1L_2G. The critical end-points lie close together. The great difference between the systems consists only in the situation of the quadruple points. In table 5 the four systems are arranged

[1]) This can also be immediately derived from the relative situation of the three-phase lines in fig. 4*a*.

according to ascending quadruple point temperatures. The order in temperature is the same as that in pressure.

TABLE 5.

Quadruple points SL_1L_2G.

o-toluidine		m-toluidine		aniline		p-toluidine	
T	P	T	P	T	P	T	P
—7.5	27.5	6.3	39.2	18.0	52.0	31.5	70

Melting points.

m-toluidine	o-toluidine	aniline	p-toluidine
—30	—15	—7	44

This is indeed also necessary on account of the coincidence of the L_1L_2G lines.

The order of the melting-points of the pure components deviates from this only in so far that *m*-toluidine and *o*-toluidine have exchanged places. I have determined the four melting-points given in table 5 myself. That of *m*-toluidine was not known, as far as I could find out; that of *o*-toluidine agrees with the observation by KNOEVENAGEL [1]). For the melting-point of aniline we find —8° given; my value lies somewhat higher; TIMMERMANS' value lies again higher than mine [2]). I think, however, that I may conclude from the small melting-range presented by my preparation, that the substance was pure. We find 45° for the melting-point of *p*-toluidine in the handbooks; my value is lower, and agrees with HULETT's very carefully executed determination (43, 9°) [3]).

The compounds that occur in these systems, contain the components in the ratio 1 : 1. I think, therefore, that I have to consider them as carbaminic acids. These compounds were still unknown, only DITTE has evidently observed two of them in his experiments. It is clear from the limits of stability of the compounds, why they have not been found; at ordinary temperature only two of these compounds are possible (phenyl- and *p*-tolylcarbaminic acid). The first decomposes directly into liquid and gas, the second into solid *p*-toluidine and gas, when the tubes are opened.

[1]) Ber. **40**. 517. (1907). KNOEVENAGEL finds besides the melting-point of —15.5° another belonging to a metastable modification (—21°). TIMMERMANS' determination (—24.4°) may refer to this metastable modification.

[2]) TIMMERMANS. Bull. Soc. Chim. Belg. **27**. 334. (1914).

[3]) HULETT. Zeitschr. physik. Chem. **28**. 650. (1899).

At low temperature they can all exist at ordinary pressure, but the formation will be hampered by the afore-said reason that the compound can put a stop to the action of the two phases (gas and solid) on each other by separation.

It is worthy of note that evidently through the action of aromatic amines on carbonic acids free acids are formed in contrast with ammoniac and the aliphatic amines, which form salts.

Of the said carbaminic acids a few salts are known. When it is tried to obtain the free acids by double conversion with acid, they split up into carbonic acid and the free amines; only at high pressure or low temperature could the free carbaminic acids be formed. Yet it is possible, and even probable that these acids, though they are durable at the ordinary temperature only under increased pressure, occur in the liquids L_2. The assumption that for certain reactions the carbaminic acids can act as intermediate product, is therefore certainly not to be deemed impossible.

It appears from the P-T-diagrams that the quadruple point of p-tolylcarbaminic acid lies higher than that of phenylcarbaminic acid. Possibly the quadruple point of one of the xylylcarbaminic acids lies at still higher temperature. The as-o-xylidine certainly invites to further investigation, the quadruple point will probably have shifted here to higher temperature, as the melting-point lies higher than that of p-toluidine. It is possible that for this system the quadruple point has already disappeared; we should then pass to another type of binary systems; in this case the behaviour will become analogous to that of sulphuretted hydrogen-ammoniac, the particulars of which I described on an earlier occasion. [1])

When we think the quadruple point gradually removed to higher temperature, it will disappear when it coincides with the critical end-point. A gradual change is not to be realized, as the change in constitution takes place discontinuously. It is here, however, possible that by suitable choice of the homologues the displacement takes place in small leaps, and the transition of the type presented by these systems into that of sulphuretted hydrogen-ammoniac appears very clearly.

In his thesis for the doctorate BÜCHNER already pointed out the existence of such a transition for systems without compound. I shall describe the phenomena which present themselves in these transformations in a later treatise.

Delft, August 30th 1918. *Technical University.*

[1]) Thesis for the Doctorate (1909). Zeitschr. physik. Chemie. **71**. 214 and 671. 1910).

Physics. — *"Experimental Inquiry into the Nature of the Surface-Layers in the Reflection by Mercury, and into the Difference in the Optical Behaviour of Liquid and Solid Mercury"*. By Dr. J. J. HAAK and Prof. R. SISSINGH. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of September 29, 1918).

1. *Introduction.* Since in 1850 L. LORENZ [1]) advanced the supposition, that the elliptical polarisation on reflection by transparent bodies is the consequence of a gradual transition between the two adjoining media and elaborated this view theoretically, the influence of these surface layers has been more than once theoretically investigated, both in case of the reflection by transparent bodies and by metals [2]). There have, however, been made only few experimental investigations into the nature of these surface-layers and our knowledge of it is confined to more or less plausible suppositions. Great influence is always assigned to the grinding and polishing and also to the grinding and polishing material itself, with which these operations are made. An investigation by RAYLEIGH shows, however, that it is not yet possible to state in what way this influence arises [3]).

Besides, an influence of the condensed gas layers has often been supposed and examined [4]). Up to now however, attempts to demonstrate the influence of a condensed gas layer have not yet succeeded.

2. *Purpose of the research.* In order to obtain the optical constants of a metal, quite independent of the grinding and polishing and the material used for this purpose, mercury was chosen for this investigation. Both the liquid mercury and a mirror of solid mercury can be examined. It becomes then also evident, whether at the transition from liquid to solid mercury the optical constants are subjected to a modification. In the investigation of liquid mercury the impression,

[1]) L. LORENZ, Pogg. Ann., **111**, 460, 1860; **114**, 238, 1861.

[2]) C. A. VAN RIJN VAN ALKEMADE, Thesis for the doctorate, Leiden, 1882; Wied. Ann., **20**, 22, 1883. P. DRUDE, Wied. Ann., **43**, 126, 1891; R. C. MAC LAURIN, Proc. Roy. Soc., (A), **76**, 49, 1905.

[3]) RAYLEIGH, Proc. Roy. Inst., **16**, 563, 1901.

[4]) J. J. SEEBECK, Pogg. Ann., **20**, 35, 1830; P. GLAN, Wied. Ann., 11, 464, 1880; R. SISSINGH, Thesis for the doctorate, Leiden, 1885; Arch. Néerl., 20, 171, 1886.

however, gained ground, that the layers of air, which are condensed on the surface, exert an appreciable influence on the elliptical polarisation at the reflection, so that in the first place this influence has been more closely examined.

3. *The used monochromator.* The determination of the optical constants took place in an entirely analogous way, as has been described by one of us [1]). For the investigation a goniometer has been used, the graduated circle of which can be placed vertical. Before the goniometer there is a monochromator of a very simple structure, which was constructed from material, present in the laboratory. The monochromator consists of a collimator with an aperture 1 : 6, a flint-glass prism of STEINHEIL with an angle of refraction of 60°, and a second collimator, which will be referred to in future as the collimator of the goniometer. The illumination takes place by means of an arc-lamp of 18 Ampères. A lens of 7 dioptrics forms an image of the crater on the slit of the collimator of the monochromator. Behind the prism a lens of 11 dioptrics forms a spectrum on the collimator of the goniometer. The axes of the two collimators are placed horizontal, the slits and the edge of the refracting angle of the prism vertical. Care has always been taken, that the image of the crater and the spectrum fall on the middle of the collimator slits. A silvered glass mirror is adjusted to the collimator of the goniometer; it can revolve round an horizontal axis and throws a monochromatic, cylindrical beam of light at the required angle of incidence on the mirror in the middle of the goniometer. The wave-length of the incident rays lies between 5790 and 5990 Ångström-units. During the observations the invariability of this colour-sifting of the incident beam of light is repeatedly examined. The fringes in the BABINET compensator are always uncoloured. Monochromator and goniometer are mounted on a firm foundation, erected free from the floor in the room, in which the observations have been made. The three levelling-screws of the legs of the goniometer stand each on two thick pieces of india-rubber, in order to prevent as much as possible the influence of vibrations on the liquid mercury surface.

4. *The goniometer.* For the adjustment and the centring of the parts of the goniometer we refer to the investigation of one of us [2]). Only a few points are briefly mentioned here.

---

[1]) R. SISSINGH, loc. cit.

[2]) R. SISSINGH, loc. cit.

The following expedient proved very convenient in the mutual adjustment of the parts of the goniometer. The sledge on which the silver mirror stands, which served for these adjustments, was fastened to a socket, which fits over a conical pivot in the middle of the goniometer circle. This pivot can be levelled with three adjusting screws. The mirror with the socket can in this way easily be placed on the goniometer and can be removed from it [1]).

As polarizer a nicol is used with pretty large oblique end-planes, which only gives a small deviation to the rays of light, which pass through it. This amounts to 2′.5.

For the adjustment of the compensator we refer again to the investigation of one of us [2]).

The polarisation planes of *each* of the two compensator wedges are placed in the required position, i. e. parallel to the plane of incidence and normal to it [3]).

To bring the movable wedge in the required position, we make use of the images of the collimator slit, which are formed in the eye-piece behind the goniometer. There are formed three pairs of images. The images of each pair coincide, if the planes of the wedges are parallel. The principal positions of the nicols, in which their planes of polarisation are parallel to the plane of incidence or normal to it, have been determined both in the vertical and in the horizontal position of the goniometer circle. The azimuth of the polarizer is called 0, when the light, that the polarizer transmits, vibrates normal to the plane of incidence, that of the analyzer, when the direction of vibration of the transmitted light is pallallel to the plane of incidence. We obtained successively in the horizontal and vertical position of the goniometer circle:

			mean
Polarizer in azimuth 0°	81°57′,	81°43′;	81°50′
Analyzer „ „ 0°	86°33′,	86°35′;	86°34′ [4]).

Considering the inevitable errors of observation, the agreement may be called satisfactory.

---

[1]) For fuller details compare J. J. Haak, Thesis for the doctorate, Amsterdam, 1918.

[2]) R. Sissingh, loc. cit.

[3]) It is noteworthy, that it is *supposed* both in the investigation of R. Hennig, Gött. Nachr., **13**, 365, 1887, as in that of P. Drude, Wien. Ann., **34**, 489, 1888; **36**, 532, 1889; **39**, 481, 1890, that the principal sections of the wedges are normal to each other and only the angle between the principal section of one of the wedges and the plane of incidence is determined and taken into account. Cf. Sissingh in Bosscha's Textbook of physics, Light, II, p. 555, note 2.

[4]) All the experiments have been made by J. J. Haak. Compare for further details Mr. Haak's thesis for the doctorate, Amsterdam, 1918.

The positions of the movable compensator wedge, in which the difference of phase for the narrow beam of light, which passes between the threads before the fixed wedge, amounts successively to $-\frac{1}{2}, 0, +\frac{1}{2}$, are:

63.86; 49.52; 35.10.

As the displacements of the wedge are always reduced to those between 49.52 and 63.86, a displacement of 14.34 mm. corresponds with a phas-edifference of $\frac{1}{2}$ or π. [1])

The angle of incidence on the mercury surface is derived from the position, in which the line of sight of the telescope of the goniometer is horizontal. This position is found halfway between the positions, in which that line of sight runs successively parallel to that of an incident beam of light and of the corresponding beam of light, which is reflected by a liquid surface in the centre of the goniometer. These positions are 81°36′ and 104°24′, so that the axis of the telescope of the goniometer runs horizontal in the position $81°36' + (104°24' - 81°36') : 2 = 93°0'$. As reflecting surface is used that of thick machine oil, because this gives rise to a pure image of the slit, which is not the case with mercury. [2]) The goniometer circle is vertical, when the axis of the incident beam coincides with that of the tube in which later the polarizer is placed and after reflection with the axis of the telescope.

5. *Preliminary investigation of the optical influence of a condensed layer of air.* When, as was already communicated in § 2, it became evident from the observations, that the layer of air condensed on the mercury surface was optically active, the angles of principal incidence I and the principal azimuth H were determined on the mirror of pure mercury, immediately after the formation. This took place by the determination of the phase-difference and the restored azimuth for two angles on both sides of the angle of principal incidence from extensive series of observations. From these values those of I and H were obtained by interpolation. [3])

It was found that $I = 79°18'$, $H = 35°45'$.

This determination could be made within three hours.

On an earlier occasion we obtained $I = 78°23'$, $H = 36°18'$, in which the mercury surface was exposed to the air for some days,

[1]) The phase-differences are given as phase-retardations with respect to the light vector in the reflected light, perpendicular to the plane of incidence.

[2]) For the mercury surface the compensator fringes remain however straight, only slightly less sharply defined. The accuracy of the adjustment of this fringe is somewhat less than with a solid surface, viz. 0.03 instead of 0.02.

[3]) R. Sissingh, loc. cit.

but protected against contact with dust, before the observations took place. In both cases the mercury had been distilled in vacuum.

With mercury that had been purified by being shaken with potassium hydroxide and nitric acid and had then been dried, we found in the same way some days after the formation of the mercury surface:

$$I = 78°14' \quad , \quad H = 36°12',$$

which is in satisfactory agreement with the second determination. The sign of the difference between the values of I and H in the two first determinations is the same as is to be derived from the theoretical research by DRUDE. A surface layer greatly diminishes the angle of principal incidence and enlarges the principal azimuth but little [1]).

The following observations show the influence of the layer of air very clearly. The pure mercury is in a well cleaned receptacle of LEYBOLD with plane parallel side walls, as is very often used for light-filters. The observations have been made at an angle of incidence of 78°, with the analyzer at an azimuth of 45° [2]).

	Adjustment of the Compensator	Polarizer
	56.46 [3])	43°33′

Two days later, during which time the mercury was protected from dust,

	56.71	43°47′

After the mercury in the receptacle had been shaken and a new surface had been formed:

	56.44	43°33′
After one day	56.64	43°44′
After shaking	56.42	43°34′

In a following set of observations, in which a pure surface was obtained by means of the method of overflowing of RÖNTGEN [4]), which has also been used by RAYLEIGH, the observations yielded:

	Adjustment of the Compensator	Polarizer
	56.43	43°44′
After two days	56.74	43°41′

[1]) P. DRUDE, Wied. Ann., **36**, 532, 865, 1889; **43**, 126, 1891.

[2]) At this azimuth the error in the determination of the phase-difference introduced by the metallic reflexion and of the azimuth of the restored plane polarisation, which henceforth will be referred to as restored azimuth, is a minimum. Cf. R. SISSINGH, Thesis for the doctorate, p. 57; Arch. Néerl., **20**, p. 188.

[3]) These and all the further values are every time the mean of four readings.

[4]) RÖNTGEN, Wied. Ann., **46**, 152, 1892; RAYLEIGH, Phil. Mag., (5), **30**, 398, 1890.

After overflowing	56.45	43°37′
After one day	56.68	43°44′
After overflowing	56.45	43°50′

The polarizing action of the side walls of the glass receptacles not having been examined, these values have no absolute value, but they represent the change by the adsorbed air layer no doubt very accurately.

6. *Change of the thickness and the influence of the layer with the time.*

There is no doubt but the thickness of the adsorbed layer of air and so also its optical influence increases with the time. In order to examine this influence the adjustments must be effected in a short time. Those of the polarizer have been omitted, because it appears from § 5, that the influence of the layer of air on the restored azimuth is very small and about of the order of magnitude of the errors of observation. In order to be able to execute the observations in a short time, only observations at one angle of incidence, viz. the angle 79°46′, which is very near the angle of principal incidence, were made, the analyzer also always being set in the same quadrant. The determination of the phase-difference, that arises at metallic reflection between the components vibrating perpendicular to and in the plane of incidence, took place only by annulling this phase difference [1]).

The shifting of the movable compensator wedge was therefore exclusively from 49.52 → 63.86 (see § 4). In this way the errors in consequence of the deviation of the light in the polarizer and the inaccurate position of the planes of polarization of the compensator wedges continue to exist, but their influence on the slight change in the phase-difference, that is to be determined, may be considered as of the second order of magnitude. Care should, however, be particularly taken, that the incident beam of light consists always of the same part of the spectrum and keeps the same direction, i. e. always falls on the middle of the slit of the collimator of the goniometer. A slight shifting of the spectrum, which was not even so much as the height of the spectrum, already modified the compensator-reading by 0.06. This is to be ascribed to the change in the angle of incidence. In the observations the mercury was placed in a shallow iron dish, attached to the bottom of a bronze cylinder. Two side-tubes, closed by plane parallel glass plates, the axes of which lie in a same meridian plane

[1]) Sissingh, Thesis for the doctorate, p. 79; Arch. Néerl., **20**, 196, 1886.

of the cylinder, enable us to make the light strike the mercury at the required angle of incidence. The mercury is conveyed into the dish by a tube in the upper surface of the cylinder. This bronze cylinder is attached to the middle of the goniometer circle. A vertical sledge makes it possible to place the mercury surface so, that the axis of the goniometer lies in it. By means of an horizontal sledge the dish may be placed so, that the axes of the incident and the reflected beam of light pass through the middle of the glass windows. The cylinder can be exhausted and filled with air, that has been dried with calcium-chloride, sulphuric acid, potassium-hydroxide and phosphor-pentoxide.

When the position of the movable prism of the compensator, immediately after the formation of the mercury surface is called c_0, that t seconds later c, and the final position c_∞, a very plausible supposition on the increase of the thickness of the layer of air and its optical influence leads to the differential equation:

$$\frac{d(c-c_0)}{dt} = k\,(c_\infty - c)$$

So that

$$c - c_0 = (c_\infty - c_0)\,(1 - e^{-kt}).$$

In this k will be proportional to the pressure of the air. This supposition is confirmed by series of observations, made at a pressure of one and of half an atmosphere, which are graphically represented in fig. 1. There have been traced six curves for different values of k. It appears clearly from the traced lines within what limits the value of k lies for the two series of observations. As value of $c_\infty - c_0$ has been taken 0.25, viz. the change of the readings in dry air after 24 hours. When the cylinder is exhausted, no change in the compensator-readings can be demonstrated even after 8 hours. Compare the line . . ., which indicates the observations in the exhausted cylinder. Here follow the means of three series of observations in dry air for a pressure of an atmosphere.

Time	Angle of Incidence 79°46′	
0	$C_0 = 56.84$	$C - C_0$
½ hour	$C = 56.863$	0,023
1 ,,	56.89	0.05
1½ hours	56.92	0.08
2 ,,	56.918	0.078
2½ ,,	56.94	0.10
3 ,,	56.965	0.125
3½ ,,	56.975	0.135
4 ,,	56.972	0.132

4½ hours	56.97	0.13
5 „	56.992	0.152
5½ „	56.99	0.15
6 „	57.00	0.16
6½ „	57.01	0.17
7 „	57.013	0.173
7½ „	57.03	0.19
8 „	57.03	0.19

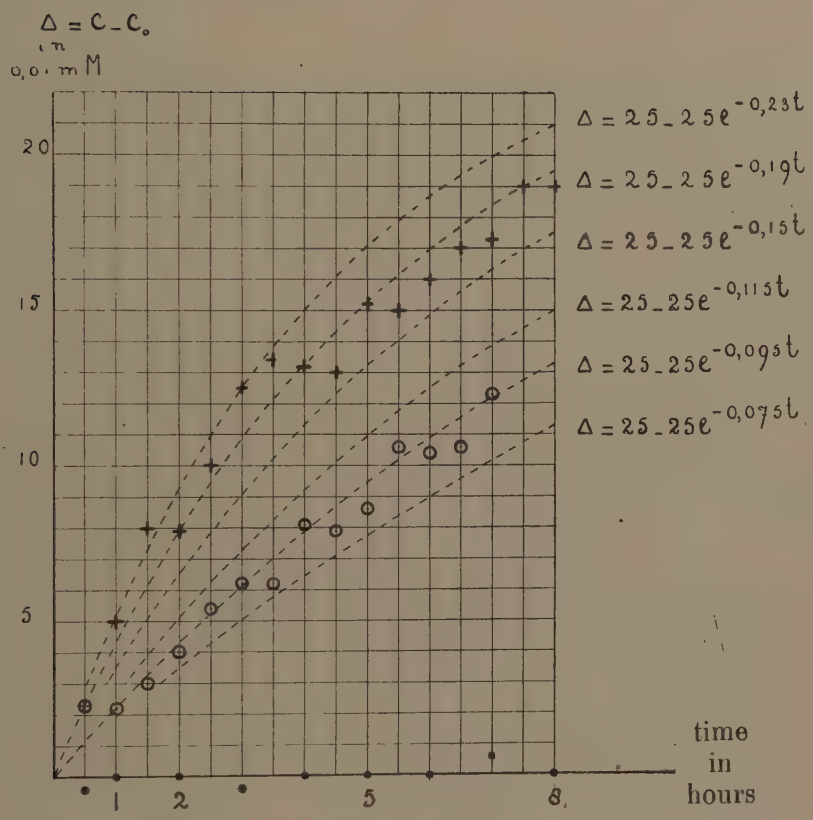


Fig. 1.

When it is borne in mind, that the error in the readings of the compensator amounts to 0.03, the agreement of the observations with the curves traced may be considered as very satisfactory. As according to § 4 a displacement of 14.34 of the movable compensator wedge corresponds to a difference of phase of π in circular measure, $c_\infty - c_0$ denotes a difference of phase of $0.25 : 28.68 = 0.0087$. In the following way the change in the angle of principal incidence I, corresponding with this, could be obtained.

The measurements on pure mercury, immediately after the formation of the mirror, gave:

Angle of Incidence	Compensator	Angle of Principal Incidence
78°38′	56.24	79°18′.

Observations made at the same angle of incidence on the mirror with an adsorbed layer of air yielded:

78°38′	56.90	78°19′.

It follows from this, that if the compensator-reading in the neighbourhood of the angle of principal incidence diminishes by 0.66, the angle of principal incidence increases by 59′, so that it follows from the observed value $c_\infty - c_0 = 0.25$, that the adsorbed layer of air decreases the angle of principal incidence by 22′.5. It is to be doubted very much whether the change, which has been observed for crystals on natural cleavage surfaces, shortly after the splitting, in the phase difference between the components of the reflected light, which vibrate normal to and in the plane of incidence, should be attributed to layers of air. DRUDE found this change for fresh cleavages surfaces of antimony glance, calcspar, and rock-salt. [1]) For rock-salt the cause is not to be looked for in a layer of water on the hygroscopical crystal. The observed change in the phase difference is greatest for antimony glance at the angle of principal incidence. According as the optical axis of the crystal is parallel to the angle of incidence or normal to it, this change amounts successively to 0.01 and 0.06. This value is many times greater than has been observed for mercury. Besides the greater part of the change has already taken place in 2 hours and the retardation in the increment of the ellipticity on standing is much more considerable than for mercury. DRUDE considers fresh cleavage-planes as unsaturate and thinks that also a greater condensation of the gas layers would have to result from this. Experiment should decide, however, whether gas layers play a part also here. Fresh cleavage-planes of lead glance do not exhibit a change in the elliptical polarisation with the time.

7. *On the changes in the phase-difference obtained in a previous investigation.*

It appears from the observations mentioned in § 5, that changes of nearly 1° in the angle of principal incidence caused by surface-layers were observed. Hence the question is, what gives rise to these greater changes? Very probably these occur in consequence of liquid layers, which are enveloped by the dust falling on the mercury. When

[1]) DRUDE, Wied. Ann., **34**, 489, 1888; **36**, 532, 1889.

some dust is swept up from the floor and some of it is strewn over the mercury, the displacement of the compensator fringe can immediately be observed without any adjustment. No liquid is, however, to be observed on the mercury either with the naked eye or with a telescope. When the dust is taken from a place, where oil has been spilt, the compensator-fringe assumes a tortuous form and liquid streaks are to be observed on the surface by means of the telescope. The sinuous compensator-fringe indicates, that not everywhere an equal quantity of liquid is spread over the mercury.

That in this case really a liquid is spread over the mercury surface, is also in agreement with the fact, that the mercury surface is smoother, so that an image may be observed in the telescope of the thread, stretched across the centre of the slit of the collimator of the goniometer. This does not succeed with a clean mercury surface in consequence of the vibrations caused by the traffic in the streets.

8. *On the values of I and H for mercury without surface layer.* It follows from § 6 that for a pressure of air of one atmosphere, the curve for $k = 0.79$ best represents the observations, so that the compensator reading successively increases by 0.045, 0.08, and 0.11 in 1, 2, and 3 hours or on an average by 0.04 an hour. As the determinations of I and H, mentioned in § 5, took up three hours, I has been diminished by 6′ on an average in this time, the principal azimuth H has, however, remained unchanged. The change in H lies namely within the errors of observation. From this follows, that for mercury without adsorbed layer of air:

$$I = 79°24' \qquad H = 35°45'$$

We subjoin the values obtained by other investigators:

BREWSTER [1])	78°27′	34°46′
QUINCKE [2])	77° 3′	33°47′
DES COUDRES [3])	79° 3′	33°30′
DRUDE [4])	79°34′	35°43′
MEYER [5])	78°23′	35°17′
MEESE [6])	79°22′	36° 7′

In this the following points are noteworthy. BREWSTER gives 26°0′ as restored azimuth after two reflections under the angle of principal incidence. From this the above given value of the principal azimuth

1) BREWSTER, Phil. Trans., 287, 1830.
2) QUINCKE, Pogg. Ann., **142**, 202, 1871.
3) DES COUDRES, Thesis for the doctorate, Berlin, 1887.
4) DRUDE, Wied. Ann., **39**, 511, 1890.
5) MEYER, Ann. d. Phys., **81**, 1017, 1913.
6) MEESE, Gött. Nachr., 530, 1913.

has been calculated. It cannot be inferred from BREWSTER's records, whether his observations have been made on a free mercury surface or on mercury against glass. In the latter case scratches in the glass may be responsible for the too low value of the principal azimuth. Also the too low value of the principal azimuth determined by QUINCKE should be attributed to scratches [1]). DES COUDRES' observations have been made on a free surface. The deviations seem to be owing to inaccuracies in the observations. DRUDE obtains a clean mercury surface by the aid of two funnels. No further particulars are given about this. The method is probably similar to RÖNTGEN's method of overflowing. The observations were made within two hours after the formation of the mercury mirror. QUINCKE and MEYER used a mercury surface against glass. A surface layer between mercury and glass has undoubtedly caused the too low value of the angle of principal incidence. MEESE also made observations on mercury against glass and demonstrates the existence of surface layers, that are then present. The values of I and H given here, have been calculated from his observations. In how far a condensed layer of air has also exerted an influence cannot be ascertained. From the values determined by us and those of DRUDE, MEESE, and MEYER, whose value for I, which is certainly too low, has not been taken into account, the following values of the optical constants for mercury may be assumed as the most probable:

$$I = 79°27' \qquad H = 35°43'.$$

§ 9. *The thickness of the adsorbed layer of air.* Both VAN RIJN VAN ALKEMADE (see § 1) and DRUDE have given equations, from which the thickness of the surface layer may be derived. According to DRUDE [2]) the change in the phase-difference between the components of the reflected light, parallel to and normal to the plane of incidence, brought about by the surface layer is:

$$\Delta' - \Delta = \frac{4\pi \, Cos\varphi \, Sin^2\varphi \, (a - Cos^2\varphi)}{\lambda \quad (a - Cos^2\varphi)^2 + a_1^2} \int_0^L \left(1 - \frac{1}{n^2}\right) dl$$

In this φ is the angle of incidence, L the thickness of the surface layer, n the index of refraction in this layer at the distance l of the reflecting metal surface:

$$a = \frac{Cos\ 4\ H}{Sin^2 I tg^2\ I} \quad , \quad a_1 = \frac{Sin\ 4\ H}{Sin^2 I tg^2\ I.}$$

[1]) J. J. HAAK, Thesis for the doctorate, Amsterdam, 1918, p. 47.

[2]) DRUDE, Wied. Ann., **39**. 481, 1890.

The index of refraction n of the surface layer will increase on approaching the reflecting metal surface. When for n a mean value is taken, viz. that between n_{air} and the value for the greatly condensed air, immediately adjoining the metal and when with QUINCKE [1]) it is supposed, that the density of this is equal to that of the mercury, $n = 4.048$ is found for this by the aid of the relation: $(n-1):d =$ constant [2]). Hence the mean value of n is $(4.048 + 1.003):2 = 2.52$. As $\Delta' - \Delta$, the phase difference for the absorbed layer of air, amounts to 0.0087 (see § 6) and $\varphi = 79°46'$ (see § 6), we find with the values of I and H mentioned in § 8:

$$L = 1.6\ \mu\mu.$$

This value is in agreement with that for the transition layer liquid-vapour, for which BAKKER [3]) gives $1-2\ \mu\mu$.

When in the same way the thickness of a layer of oil ($n = 1.5$, $d = 0.9$) is calculated, which according to § 5 can modify the angle of principal incidence by 1°, the compensator-reading by 0.50 and more, we find, introducing the value 0.50, $L = 3\ \mu\mu$. This is in accordance with RAYLEIGH's and FISCHER's [4]) determinations. RAYLEIGH found, namely, for the thickness of the thinnest layer of oil, that stops the movements of the camphor particles on water, $2\mu\mu$. FISCHER found for liquid layers, which spread over mercury, thicknesses smaller than $5\mu\mu$.

As the adsorbed layer of air of a thickness of 1.6 $\mu\mu$ changes the compensator-reading by 0.25 and the mean error in the reading amounts to 0.02, a layer of a thickness of 0.13 $\mu\mu$ can still be demonstrated in this way by this optical method. Such a layer is of the thickness of a molecule. It is not possible to prove the existence of such thin layers by the aid of the capillary phenomena.

It is not possible to remove the once adsorbed layer of air by means of a very far exhausted vacuum, as the mercury airpump of GAEDE can bring about. After eight hours' pumping no displacement of the compensator reading could be demonstrated [5]).

---

[1]) QUINCKE, Pogg. Ann., **108**, 326, 1859.

[2]) L LORENZ—H. A. LORENTZ's formula cannot be applied here, as n^2 would become negative. Let $\frac{n_1^2+2}{n_1^2-1}d_1$ be $= \frac{n^2+2}{n^2-1}d$, then $\frac{n^2+2}{n^2-1}\frac{d}{d_1}$ must be > 1, if n_1^2 is to be positive. In the case considered here $\frac{d_1}{d} = 10^4$, $\frac{n^2+2}{n^2-1} = 5.10^3$, so that the condition is not fulfilled.

[3]) G. BAKKER, Z. f. Phys. Chem., **91**, 571, 1916.

[4]) RAYLEIGH, Phil. Mag., (5), **30**, 396, 1890; FISCHER, Wied. Ann., **68**, 436, 1899.

[5]) This result is in agreement with other experiments, which show with how

10. *Testing of the obtained results by means of a mercury mirror got by distillation of pure mercury.* According to § 6 the phase-difference was measured with the compensator by annulling this phase-difference, in order to determine the influence exerted by the adsorbed layer of air. From the values given in § 4 it appears, that the lowest compensator-reading corresponds with the smallest phase-difference. As according to DRUDE every surface layer diminishes the angle of principal incidence, hence increases the phase-difference for every angle of incidence, the mercury surface is the better, i.e. less contaminated by surface layers, as the compensator-reading is smaller. For an angle, somewhat greater than *I*, this smallest reading was 56.84 [1]). Mercury, distilled in vacuum, which was conveyed into a dish after being filtered through a paper funnel, yields the reading 56.84. This dish had been placed free in the air and was not surrounded by a case (see § 6), so that the rays of light need not pass through glass windows. Mercury purified by being shaken with potassium hydroxide and nitric acid, but not distilled in vacuum, yields the reading 56.85. The difference with the preceding value falls within the errors of observation. Mercury, conveyed into the dish through a drawn-out glass tube, yields 57.00, if the tube has not been very well cleaned. For a thorough cleaning heating to a dull red glow is generally sufficient. Touched by a piece of cloth, which is not clean, the mercury gives the adjustment 57.20. When breathed upon, the mercury yields a reading increased by 0.04 or 0.05.

In order to prove by another way, that 56.84 is the reading for pure mercury without surface layer, pure mercury was distilled in vacuum into the iron dish, which is situated in the bronze cylinder. It was previously ascertained, that the glass windows — carefully cooled glass plates of a thickness of 3 mm. — did not modify the compensator-reading, even though the cylinder was exhausted of air. For this purpose an iron mirror was placed in the dish and the compensator-reading was observed before and after the exhaustion. The pure mercury was heated in a glass globe. The vapour was condensed in the spiral windings of a glass cooler and received in a glass bottle, from which it flows out into the iron vessel through a glass tube with drawn-out point. All the junctions of this apparatus consist of sealed glass. The air-tight connection of the glass tube with

great a force these adsorbed layers of air are attached to the surfaces. Cf. among others VOIGT, Wied. Ann., **19**, 39, 1884. Likewise it is in agreement with the fact, that the layer of air cannot be removed by means of carbon powder, which has been heated just before to ared glow. Cf. among others SISSINGH, Thesis for the doctorate, p. 162; Arch. Néerl. **20**, 228, 1886.

[1]) The values recorded here are again the means of four readings.

drawn-out point, which passes through the short tube in the upper wall of the bronze cylinder, with this short tube was effected by means of an airpump tube and a little collodion. A GAEDE mercury airpump exhausts the space. The compensator-adjustment was 56.84, hence exactly the same as that which prevails for mercury without adhering layer of air, according to the observations communicated in this §.

11. *Inquiry into a difference in the optical constants of liquid and solid mercury.* At first the air in the bronze cylinder was very carefully dried. The cooling of the cylinder and the dish with mercury in it took place by putting a mixture of solid carbonic acid and ether on a tin plate screwed on to the bottom of the cylindrical case. By means of ebonite as heat-insulator the conduction of heat from the metallic parts of the goniometer to the cylinder is prevented. In order to prevent the cooling of the glass windows in the side tubes, because no water vapour from the air may settle on them, these windows are cemented to ebonite tubes, which are screwed on to the side tubes of the bronze cylinder. The air-tight closure was effected by means of very tough Ramsay-grease. As it appeared that in spite of this precaution some water-vapour deposits on the glasses, a current of dry air was blown along them by means of a GAEDE box pump, which quite remedied this evil.

During the cooling the following phenomena are observed. When during the cooling the compensator is adjusted for the dark compensator-fringe, this winds, while the mercury is still liquid and becomes less dark, after which it disappears altogether. At last the fringe, which has then become black again, jumps back to its original position. When the telescope, which is placed behind the analyzer, is then adjusted on the mercury, ice crystals appear to float on the mercury. From this it is evident, that the explanation of the observed phenomenon is the following. In spite of the careful drying the air contains traces of water vapour, which are deposited on the mercury surface during the cooling and spread over it as a liquid [1]). The compensator-fringe is probably sinuous, when the water layer consists of incoherent patches and is very thin. When the water forms a coherent layer and this layer has become so thick, that the reflection takes place on water, the fringe disappears, as at the chosen angle of incidence of 79° 46′ water does not perceptibly polarize the light elliptically on reflection. As soon as the tempera-

[1]) These phenomena are not observed on an iron mirror during the cooling, so that the condensed water vapour does not seem to spread over this mirror.

ture falls below 0°, the water freezes and contracts to a few ice crystals, which lie spread with large interstices on the mercury. Then we have again a mercury surface and the normal adjustment of the compensator-fringe. In the meantime the mercury is still liquid.

Attempts to prevent this deposition of the water vapour by careful drying of the air proved unsuccessful. This was done thoroughly however by exhausting the bronze cylinder. When an iron mirror is placed in the cylinder, it appears, that the glasses on the side-tubes do not become bi-refringent when the case is exhausted, but that they do so through the one-sided cooling during the fall of the temperature of the bronze cylinder by solid carbonic acid and ether. Then the adjustment of the compensator-fringe changes by the constant amount 0.08. During the cooling of the mercury in vacuum no sudden change in the position of the compensator-fringe is observed at the moment of the freezing. The reading of the compensator-fringe diminishes gradually, till this change reaches an amount of 0.08. It appears from this that only the slight double refraction of the glasses plays a part here and that neither during the freezing of the mercury, nor during the further cooling down to —80° the phase-difference, hence also the angle of principal incidence of the mercury changes perceptibly.

At first the position of the polarizer presented a change during the freezing, which could amount to as much as 3°. This, however, must be attributed to the influence of wrinkles, which make their appearance with the freezing in consequence of inevitable vibrations caused by the traffic in the streets or by tram cars. This is in agreement with the observations of FIZEAU, DRUDE, and HAAK [1]) on the diminution of the restored azimuth through grooves or scratches in various directions on the reflecting surface. In order to prevent these wrinkles as much as possible, the iron dish is filled brimful with mercury and then the temperature is slowly lowered. When the traffic in the streets is not too great, it is then sometimes possible to get such a smooth mercury mirror, that a pure image of the collimator slit can be observed. In this case the adjustment of the polarizer does not change. It appears from this, that on freezing and cooling of the solid mercury to — 80° neither the angle of principal incidence nor the principal azimuth are subjected to any change. Optically liquid and solid mercury behave in the same way [2]),

[1]) FIZEAU, Ann. de Chim. et de Phys., **3**, 373, 1861; DRUDE, Wied. Ann., **39**, 497, 1890; J. J. HAAK, Thesis for the doctorate, Amsterdam, 1918.

[2]) Besides it follows from this that LUMMER and SORGE's supposition (Ann. der Phys., **31**, 325, 1910), according to which internal tensions would give rise to the elliptical polarisation, cannot be valid for solid mercury.

Chemistry. — "*Investigations on* PASTEUR'S *Principle concerning the Relation between Molecular and Crystallonomical Dissymmetry: VIII. On the spontaneous Fission of racemic Potassium-Cobalti-Oxalate into its optically-active Antipodes.*" By Prof. F. M. JAEGER and WILLIAM THOMAS. B. Sc.

(Communicated in the meeting of Nov. 30, 1918).

§ 1. As a continuation of the fissions, accomplished up to this date, of the racemic complex *trioxalates* of *potassium* and the trivalent metals: *chromium* [1]), *rhodium* [2]), and *iridium* [3]), — it appeared desirable to make an attempt to separate the analogously built *potassium-cobalti-oxalate*: $K_3\{Co(C_2O_4)_3\} + 3\frac{1}{2}H_2O$, into its optically-active components for the purpose of a comparison of their rotatory dispersion and crystal-forms. The series of the complex *oxalates* investigated, would then be really complete. The proposed separation into its antipodes was, however, hindered till now by a number of difficulties of various kinds, partially caused by the salt being not very resistant towards an increase of temperature, and on the other hand by its particular solubility-relations, when combined with active bases; moreover a troublesome circumstance was its sensitiveness to light-radiation, this causing a rapid decomposition of these salts in solution, under formation of a pale pink, hardly soluble precipitate, — a reaction, the study of which is now started in our laboratory.

The racemic salt: $K_3\{Co(C_2O_4)_3\} + 3\frac{1}{2}H_2O$ has been studied by COPAUX [4]). It has, like the corresponding salts of the other metals, triclinic symmetry, but it is *not* isomorphous with them, as follows already from the deviating content of water of crystallisation: while the *iridium*- and *rhodium*-salts contain $4\frac{1}{2}$ molecules of water, the corresponding *chromi*-salt has *three*, the *cobalti*-salt $3\frac{1}{2}$ molecules of it. The salt was prepared in the following way in greater quantities.

A mixture of 25 grammes of *cobalti-carbonate*, 250 ccm of a satu-

---

[1]) A. WERNER, Ber. d. d. Chem. Ges. **45**. 3061. (1912).

[2]) A. WERNER, Ber. d. d. Chem. Ges. **47**. 1954. (1914); F. M. JAEGER, Proceed. Ak. v. Wet. Amsterdam, **20**. 263. (1917).

[3]) F. M. JAEGER, Proceed. Kon. Akad. v. Wet. Amsterdam, **20**. 273. (1917); **21**. 203. (1918).

[4]) H. COPAUX, Bull. de la Soc Min. **29**. 75. (1906); Ann. de Chim. et Phys. (8). **6**. 508. (1905).

rated solution of *potassium-oxalate*, and 230 ccm of a saturated solution of *oxalic acid*, was heated on the waterbath under continuous stirring, till all the carbonate had entered into solution. The liquid thus obtained was cooled to 40° C, and then 30 grammes of finely pulverised *lead-peroxide*: PbO_2, were added. After some time 50 ccm of a 50 % solution of *acetic acid* were slowly added to the rigorously cooled solution under continuous stirring. Then the liquid was filtered and precipitated by 400 ccm of 97 % alcohol; the green precipitate was sucked off, and several times washed with absolute alcohol. In this way 80 grammes of the dark green *potassium-cobalti-oxalate* were readily collected.

§ 2. The *silver*-, and the *barium*-salts being both only slightly soluble, we used the *potassium*-salt itself for the preparation of the corresponding *strychnine*-compound, thus avoiding the troublesome use of large volumes of solution; this is of advantage, because also at lower temperatures the solutions are partially decomposed under development of carbondioxide. The *strychnine*-salt [1]) is for the greater part precipitated, if the calculated amount of *strychnine-sulphate* is added to the solution of the *potassium*-salt, and only so much cold water is consequently added as to dissolve the precipitate formed. All these experiments were executed in a dark room, where the solution is left standing in an open vessel for several weeks, at a temperature of about 16° C; the fractions successively deposed from the mother-liquor are collected separately.

The crystals obtained are treated, in the same way as described on former occasions, with an excess of *potassium-iodide*, the *strychnine-iodide* is sucked off, and the filtrate precipitated by means of 97 % alcohol. The salt obtained is purified by repeated crystallisations from a small quantity of water.

The first fractions of the *strychnine*-salt in this way gave crystals of the *laevo*gyratory antipode, containing 1 molecule of water of crystallisation. The determination of the water-content cannot be made at 120° C, because of the decomposibility of the substance; it was therefore made by passing a current of dry air at 20° C over the finely powdered substance during a very long time, and a loss of weight corresponding to 0,8 molecules of water was finally observed.

With respect to the light-absorption by the dark green solutions,

[1]) Originally the separation of the racemic salt was tried by the aid of *cinchonine*, but without success. Afterwards we repeated these experiments under somewhat varied conditions, but they gave no positive results either.

it was found that in layers of 20 cm., a solution of 0,41 percentages by weight showed a pronounced absorption-band in the yellow and blue part of the spectrum between the wavelengths of 5510 and 6520 *A.U.* For concentrations of 0,82% and 1% no light was transmitted; but the solution just mentioned allowed the light to come through between 4850 and 5515 A.U. With a solution of 0.31% these limits were: 4770 to 5670 A.U., and 6480 A.U.; with one of 0.27%: 4720 to 5750, and 6450 A.U.; etc. Determinations corresponding to wave-lengths within these limits can only be made with extremely diluted solutions, and the incertitude of the readings caused thereby may explain the deviations of the values obtained in the case of the laevo-, and dextrogyratory components, in so far as these values are observed in the immediate vicinity of the deep minimum in the dispersion-curve. But notwithstanding this incertitude, the characteristic slope of the dispersion-curve is in all cases fixed with full certainty.

For the salt from the first fractions, we found values of the rotation in good agreement with each other, which are suited to elucidate the strange form of the dispersion-curve (fig. 1) immediately:

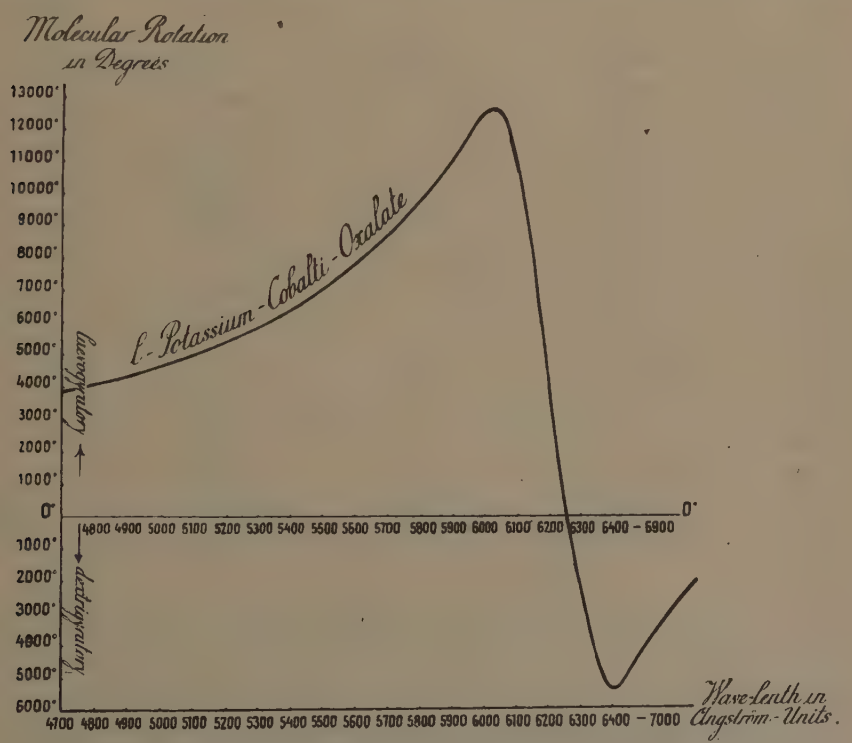


Fig. 1. *Molecular Rotatory Dispersion of Laevogyratory Potassium Cobalti-Oxalate* ($+1\,H_2O$).

Molecular Rotation of Laevogyratory Potassium-Cobalti-Oxalate (+ 1 H_2O).	
Wave-length in Angström-Units:	Molecular Rotation in Degrees:
4730	— 3913°
4780	4031
4870	4196
4945	4399
5020	4619
5105	4916
5180	5123
5260	5487
5340	5900
5420	6387
5515	7086
5610	7805
5700	8682
5800	9708
5910	11327
6020	12508
6140	— 8506
6260	+ 263
6380	5391
6520	4126
6660	1799
6800	+ 160

This curve therefore appears to possess the peculiarity, that the rotation at first rapidly increases for greater wave-lengths, but decreases then very steeply in the vicinity of the absorption-band, to assume the opposite algebraic sign at circa 6260 A.U. The right-handed rotation now reached, shows a minimum at about 6400 A.U., and decreases at first rapidly, afterwards more slowly, so that the

curve approaches more and more the axis of the *zero*-value. The dispersion has therefore, properly speaking, only a really "normal" character between 6240 A.U., and 6400 A.U. The maximum of the laevogyratory rotation is situated at about 6000 A.U.

On comparing the magnitude of the rotations for corresponding wave-lengths, in the cases of the complex *oxalates* of *chromium*, *cobaltum*, *rhodium*, and *iridium*, — the influence of the specific nature of the central metallic atom on the whole character, as well as on the absolute values of the rotation, is immediately evident. The figures 2 and 3 will show this clearly; in fig. 2 the curves of the *cobalti*-, *rhodium*- and *iridium*-salts are drawn, in fig. 3 those of the *chromium*-, and *cobalti*-salts. While the complex *rhodium*-, and *iridium-oxalates* show an analogous dispersion, the *cobalti*-salt seems to have a

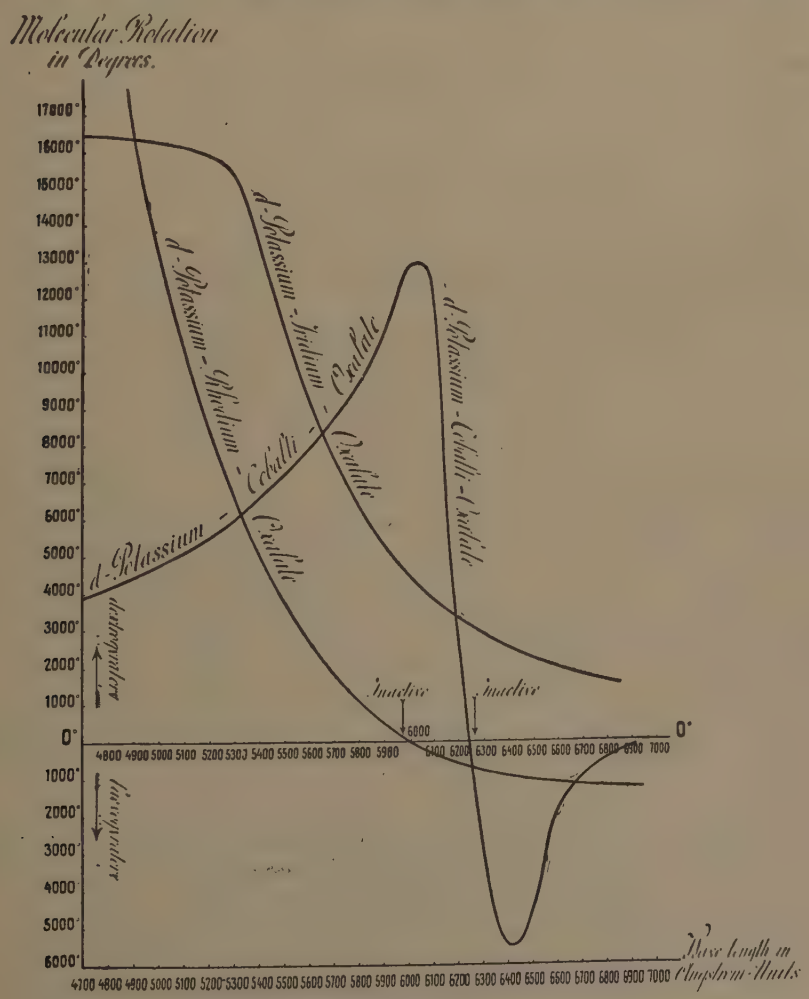
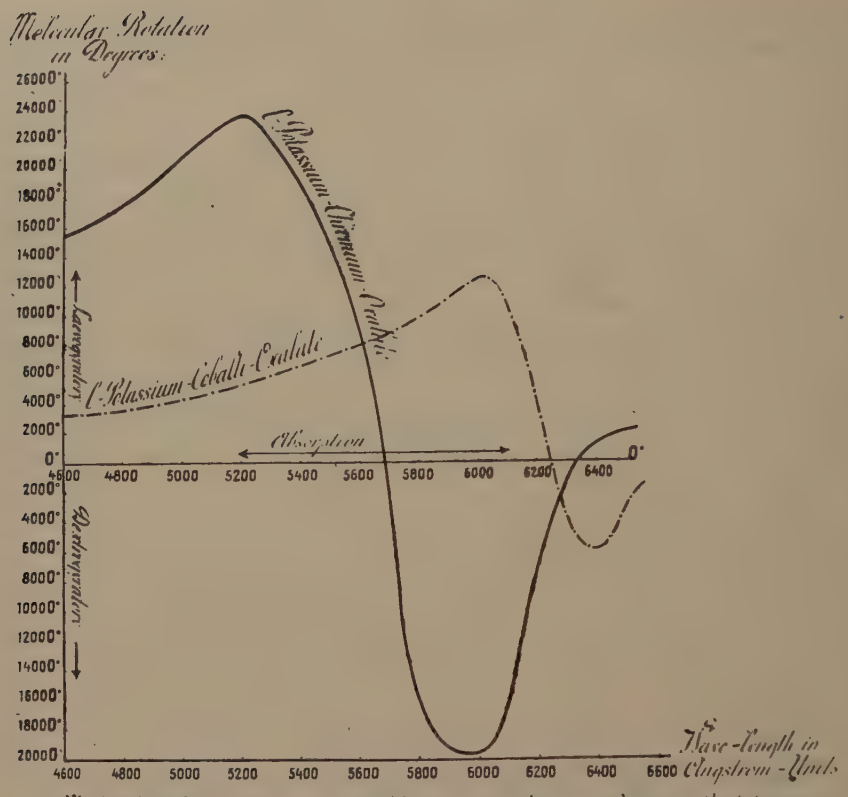


Fig. 2.



Molecular Rotation-Dispersion of Laevogyrotory-Potassium-Chromium Oxalate ($+3H_2O$) and Potassium-Cobalti-Oxalate ($+1H_2O$)

Fig. 3.

deviating position amongst them. [1]) However, this salt shows a complete analogy in this respect with the corresponding *chromium*-salt, as may be easily seen from the measurements made in this laboratory by Mr. P. J. BECKER with the *potassium-chromi-oxalate:* $K_3\{Cr(C_2O_4)_3\} + 3H_2O$, which was separated into its antipodes after WERNER's method [2]). On comparing the rotation of this salt with that of the *cobalti*-salt at corresponding wave-lengths, it may be seen that the rotations of the *chromium*-salt are, up to 5640 A.U., and above

[1]) According to an investigation of G. BRUHAT (Bull. de la Soc. Chim. (4), **17**, 226 (1915), there is also a maximum in the dispersion-curve of the complex *iridium*-oxalate at short wave-lengths (about 4930 A.U.). It therefore seems probable, that the character of the anomalous dispersion curves is really very analogous in all these cases, however with a considerable difference in the positions of the maxima.

[2]) A. WERNER, Ber. d. d. Chem. Ges. **45**. 3061 (1912). It is difficult to obtain good crystals of this compound which, moreover, rapidly autoracemises in solution; therefore crystallographic measurements could till now not be made in any way.

6280 A.U., considerably *greater* than those of the *cobalti*-salt; while between 5640 and 6280 A.U. the reverse occurs, and the two salts even show an opposite rotation as a consequence of the shift of their maximum and minimum; this shift is, in its turn, intimately connected with the very different situation of the absorption-bands: for the *chromium*-salt has a very broad band in the orange, yellow, partially in the green and the violet. It must, moreover, be remarked, that our former experience has made it clear, that the *triethylene-diamine-chromi*-salts show in general only about half the rotations of the *triethylenediamine-cobalti*-salts for corresponding wave-lengths. From this it is evident how great and unexpected an influence the presence of *basic* groups, or that of *acid* radicles, has, on the magnitude of the rotation when they are dissymmetrically arranged round the central metal-atoms of such complex salts.

§ **3**. As was already mentioned before, the racemic *cobalti*-salt has already been investigated by COPAUX, who described it as a *triclinic* substance, but *not* isomorphous with the corresponding *rhodium*-salt. As will be soon demonstrated, COPAUX most probably obtained his crystals from solutions evaporated in the darkness and at low temperatures; for even in diffuse day-light the solution is decomposed with development of carbondioxide and precipitation of *cobalto-oxalate*, — a decomposition which is quite analogous to the photochemical decomposition of the corresponding *ferric*-salt. It must be remarked that in this photochemical reaction, *blue* light decomposes the solutions much more rapidly, than *red* or *green* light, — in full agreement with DRAPER's law. But it is a remarkable fact that *yellow* and *orange* light, which is absorbed also to a considerable amount, has scarcely any stronger influence than the only slightly active green or red rays.

In our experiments the saturated solutions were evaporated in a dark space, the temperature of which differed only slightly from 0° C. In this case we really obtained *triclinic-pinacoidal* crystals of a dark green, almost black colour, and showing in most cases curved faces and rudimentary forms; accurate measurements were therefore very difficult. The angular values obtained really differ not inconsiderably from those published by COPAUX, at least within some zones; but the identity of his crystals and ours need not be doubted in any way, as may appear from the following values:

Angular values:	*Observed:*		*Calculated:*
	JAEGER:	COPAUX:	
$m : \mu = (110) : (1\bar{1}0) =$	60° 37′	60° 36′	—
$c : m = (001) : (110) =$	79 22	79 26	—

$q:m = (011):(110) =$	65	10	65	28	—
$a:b = (100):(0\bar{1}0) =$	89	2	88	42	88° 40′
$c:q = (001):(011) =$	33	10	32	28	—
$b:c = (010):(001) =$	88	37	88	39	88 32

etc.

The specific gravity of these crystals, which commonly only showed the combination-forms: $m = \{110\}$, $\mu = \{1\bar{1}0\}$, $c = \{001\}$, $q = \{011\}$, $\omega = \{\bar{1}12\}$, and $a = \{100\}$ and $b = \{010\}$ very narrow, — was at 15° C. determined at: $d_{4^\circ} = 1{,}877$; the molecular volume is therefore: 268,14.

An analogy of form with the corresponding *rhodium*-, and *iridium*-salts is not present; neither is this the case with the monoclinic *chromium*-salt, which possesses $3\,H_2O$. [1]).

These facts prove in every case undeniably, that at temperatures in the neighbourhood of 0° C., the saturated solutions deposit crystals of the *racemic* compound. But the stranger therefore appeared to us originally the behaviour of solutions evaporating at room-temperature. For from an also inactive solution, which during the summermonths was slowly evaporated in a dark room at temperatures only slightly differing from 18° C., dark green, almost black needles were obtained, which even on superficial examination appeared to differ appreciably from the triclinic racemate. Crystal-measurements taught us, that they had *trigonal* symmetry, and that their form was identical with that of the laevogyratory antipode. A crop of small crystals of this crystallisation-product, dissolved in water, did however not show any appreciable rotation. Suspicion immediately arose, that the racemic salt might have been split under these circumstances into its antipodes spontaneously, and that no trace of rotation could be detected according to our way of investigation, only because the solution deposes the crystals of the two antipodes besides each other in about equal number, so that a crop of several crystals, which by the lack of hemihedral faces cannot be discerned from each other, contains in general almost an equal number of dextro- and laevogyratory individuals, when collected from the solution at random. It is evident that such a mixture will not exhibit any appreciable optical activity. If this suspicion were true, the optical activity must appear immediately, if only a single crystal at the same time were dissolved. Indeed, experience proved,

---

[1]) The parametres of the *cobalti*-salt are: $a:b:c = 0{,}5963:1:0{,}6590$; $\alpha = 91°42'$; $\beta = 101°23'$; $\gamma = 88°22'$. The *chromium*-salt is monoclinic, with: $a:b:c = 1{,}0060:1:1{,}3989$; $\beta = 86°0'$. For the *rhodium*- and *iridium*-salts, cf. these Proceedings, **20**, p. 270, (1917); and **21**, 214, (1918).

that the first crystal thus investigated, showed the full activity of the *dextro*gyratory component, which then had not yet been obtained in the pure state by our fission-experiments; the following numbers can convincingly demonstrate the fact mentioned:

Molecular Rotation of the Dextrogyratory Potassium-Cobalti-Oxalate (+ 1 H_2O).

Wave-length in Angström-Units:	Molecular Rotation in Degrees:
4730	+ 3876°
4780	4009
4870	4167
4945	4428
5020	4689
5105	4923
5180	5106
5260	5553
5340	6013
5420	6416
5510	7023
5610	7916
5700	8703
5800	9764
5910	11365
6020	12812
6140	+ 8269
6260	— 103
6380	5468
6520	4317
6660	1678
6800	526
6940	— 198

The minimum in the dispersion-curve is here somewhat steeper

than our former measurements with the laevogyratory component indicate, — a deviation probably caused by the uncertainty of the readings in the interval of the absorption-band. But the correspondence of the curves cannot be doubted any longer. Continued study of the single crystals deposited from the solution, taught, that besides the lefthanded crystals, also those of the laevogyratory component occur. It is commonly quite impossible to recognise the two kinds of crystals from each other by their outward appearance, and thus to select them, because the facets of the right- or lefthanded trapezohedra or trigonal bipyramids are commonly lacking, so that the aspect of the crystals is in both cases quite the same.

It cannot be doubted therefore any longer, that we have found here a *first* instance of a fission into optically active antipodes of such complex metallic compounds, *by spontaneous crystallisation*; for the case of *potassium-rhodium-oxalate* formerly indicated by WERNER as an instance of this kind, can no longer be considered as such, as was some time ago proved by us.[1])

For the purpose of justifying this view, it was necessary to determine the *temperature of transition* of the racemic compound into its antipodes as accurately as possible. This was done in two ways: by means of the dilatometrical method, and by the study of the

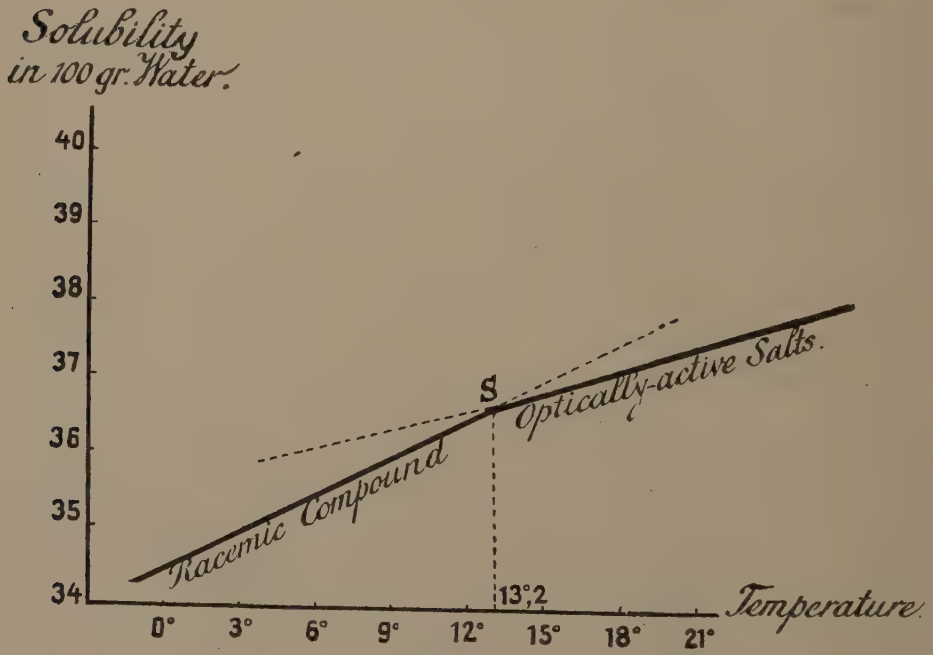


Fig. 4.

[1]) F. M. JAEGER. Proceed. Kon. Akad. v. Wet. Amsterdam, **20**. 264. 265. (1917).

solubilities of the salts at different temperatures. For we have already formerly drawn attention [1]) to the fact, that below and above the transition-temperature, the metastable forms must have, as in all such cases, the greater solubility; and this was exactly one of the arguments used by us to reject WERNER's conclusions about the occurrence of a spontaneous fission in the case of the *potassium-rhodium-oxalate*.

Indeed, our experiments fully confirmed this view: we were able to demonstrate, that below 14° C. the solubility of the inactive form is really *smaller* than that of the optically-active antipodes, whereas above 14° C. the reverse was the case. Thus 100 grammes of water at 0° C. e.g. appeared to dissolve 34,50 grammes of the racemic salt, at 14° C. 36,81 grammes; etc. On the other hand, 100 grammes of water at 20° C. dissolved 37.40 grammes of the laevogyratory salt, at 22° C. 37,6 grammes; etc. The fig. 4 shows, that the transitionpoint to be determined, without appreciable error, may be fixed as 13,2° C.; this temperature, at which the reaction:

$$2\,rac.\,[K_3\{Co(C_2O_4)_3\} + 3\tfrac{1}{2}H_2O] \overset{13°,2}{\rightleftarrows} [d\text{—}K_3\{Co(C_2O_4)_3\} + 1H_2O] + [l\text{—}K_3\{Co(C_2O_4)_3\} + 1\,H_2O] + 5H_2O,$$

takes place, is therefore a *minimum*-temperature for the existence of the optically-active salts.

The dilatometrical experiments were rather difficult, because of the tendency of the compound to decompose, when its solution is kept at somewhat higher temperatures for a long time, and because of the inevitable retardation-phenomena. Notwithstanding this, we were able to prove a sharp discontinuity of the volume-temperature-curve, at a temperature between 12° and 16° C. That such retardation-phenomena really occur, cannot be doubted; even in solution, the active salt is transformed just below the transition-temperature into the racemic one, with considerable slowness. Thus we found, that at 12° C, the dilute solution of the laevogyratory antipode lost in one day about half, in two days two thirds, in three days almost five sixths, and in four days about nine tenths of its original optical activity, while at the said temperature the optical antipodes beyond all doubt are already metastable with respect to the racemic salt.

Crystallisation-experiments made in a thermostate at 22° C gave results in full agreement with our conclusions: the solutions deposited

[1]) A. WERNER, Ber. d. d. Chem. Ges. **47**. 1954 (1914).

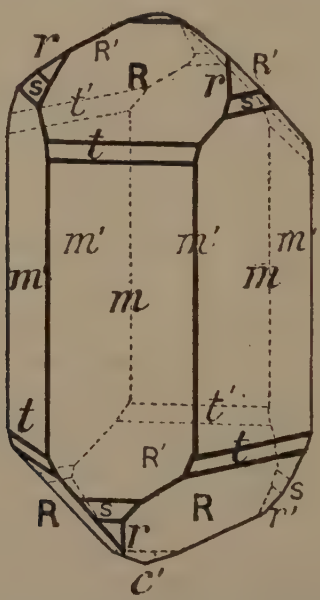
always the trigonal needles of the active components besides each other, but at 0° C we got only the triclinic crystals of the racemate. Therefore complete proof has been given now, that a fission of the *potassium-cobalti-oxalate* by spontaneous crystallisation into its optically active components really occurs at temperatures above 13°,2 C.

§ 4. Crystallographical research taught us, that both the optically active components occur in commonly not distinguishable crystals of *trigonal-trapezohedrical* symmetry, which are completely *isomorphous* with those of the optically-active *rhodium-*, and *iridium*-salts.

They have the appearance (fig. 5 and 6) of a prismatic forms of more or less extension; the *dextro*gyratory component hitherto always presented the rhombohedron-like shape of fig. 6.

Trigonal-trapezohedrical.

$a : c = 1 : 0{,}8968$ (BRAVAIS); $\alpha = 100°27'$ (MILLER).

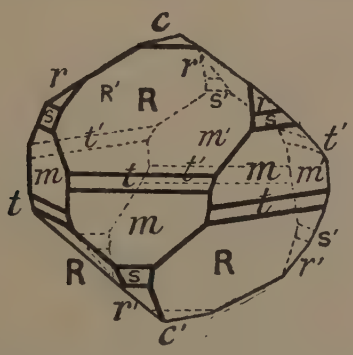


Laevogyratory Potassium-Cobalti-Oxalate $(+ 1H_2O)$

Fig 5.

Forms observed: $R = \{10\bar{1}1\}$ $[100]$, large and very brilliant; $c = \{0001\}$ $[111]$, always present, but subordinate; $m = \{10\bar{1}0\}$ $[2\,\bar{1}\,\bar{1}]$, commonly predominant with the lefthanded crystals, and in the case of the dextrogyratory individuals small, but well developed and yielding sharp images; $r = \{01\bar{1}1\}$ $[221]$, and $s = \{02\bar{2}1\}$ $[11\bar{1}]$, always present, small, but very lustrous; $t = \{20\bar{2}1\}$ $[5\,\bar{1}\,\bar{1}]$, rather large and yielding good reflections. Hemihedral combination-forms were hardly ever

observed; only once a crystal of the *laevo*gyratory antipode presented the *right*handed trigonal bipyramid $x = \{22\bar{4}1\}\,[71\bar{5}]$ as an extremely narrow truncation of the edge $R : m$. From this it may be concluded, that in this case too, the substance manifests only a very weak tendency to present hemihedral forms, unregarded the enormous optical activity, which these salts exhibit in aqueous solution; and, moreover, that also in the case of the complex *oxalates*, the same morphological relation between the *cobalti-* and the *rhodium*-salts appears to exist, as between the corresponding *cobalti-* and *rhodium-triethylenediamine-nitrates* [1]), in so far, as the *oppositely* rotating *cobalti-* and *rhodium*-salts, which are separated from the less soluble compounds with optically-active bases (*strychnine*) or acids (*tartaric acid*), yet exhibit hemihedral forms of *the same* algebraic sign:



Dextrogyratory Potassium-Cobalti-Oxalate $(+ 1H,O)$.
Fig 6.

Substance:	Rotation of the salt separated from the less soluble compound:	Algebraic sign of the hemihedral forms present:
Triethylenediamine-Cobalti-Nitrate.	*(Chloro-tartrate)*: **d.**	+ sphenoid.
Triethylenediamine-Rhodium-Nitrate.	*(Chloro-tartrate)*: **l.**	+ sphenoid.
Potassium-Cobalti-Oxalate.	*(Strychnine salt)*: **l.**	+ bipyramid.
Potassium-Rhodium-Oxalate.	*(Strychnine-salt)*: **d.**	+ bipyramid.
Potassium-Iridium-Oxalate.	*(Strychnine-salt)*: **d.**	+ bipyramid.

At the same time it is evident that the *cobalti-* and *rhodium*-salts

[1]) F. M. JAEGER, Proceed. Kon. Akad. v. Wet. Amsterdam, **20**. 258, 261 (1917).

set free from the less soluble compounds with optically active bases or acids, in the case of *triethylenediamine*-derivatives exhibit just the *opposite* rotatory power, as is observed in the case of the complex *trioxalates*; a fact, which sustains the view, according to which it is the basic or acid nature of the radicles placed round the central metallic atom, rather than the special nature of the latter, which in the first instance determines the direction of the rotation.

Angular Values:	*Observed*:	*Calculated*:
$c:R = (0001):(10\bar{1}1) =$	* 46° 5′	—
$m:t = (10\bar{1}0):(02\bar{2}1) =$	25 43	25° 46½
$m:m = (10\bar{1}0):(01\bar{1}0) =$	60 0	60 0
$R:R = (10\bar{1}1):(\bar{1}101) =$	77 11	77 12
$c:r = (0001):(01\bar{1}1) =$	46 8	46 5
$r:s = (01\bar{1}1):(02\bar{2}1) =$	18 0	18 8½
$s:m = (02\bar{2}1):(01\bar{1}0) =$	25 55	25 46½
$s:R = (02\bar{2}1):(10\bar{1}1) =$	51 13	51 5
$R:t = (01\bar{1}1):(02\bar{2}1) =$	17 58	18 8½
$R:m = (10\bar{1}1):(10\bar{1}0) =$	43 55	43 55
$x:R = (22\bar{4}1):(01\bar{1}1) =$	78 13	78 9

No distinct cleavage was observed.

The specific gravity of these crystals was at 15° C. determined at: $d_{4^0} = 1{,}8893$; the molecular volume is therefore: 242,57, and the topical parameters become: $\chi : \omega = 7{,}4676 : 6{,}6971$; or $\chi' = 6{,}3789$. The values of χ of the *Co*-, *Rho*-, and *Ir*-compounds appear to decrease therefore continuously with increasing atomic weight of the metallic atom, while ω reaches a minimum in the case of the *Rho*-salt.

Groningen, November 1918.

Laboratory for Inorganic and Physical Chemistry of the University.

Mathematics. — *"Ueber eineindeutige, stetige Transformationen von Flächen in sich"* (sechste Mitteilung [1]). By Prof. L. E. J. BROUWER.

(Communicated in the meeting of November 30, 1918).

§ 1. In einem in 1912 in den Göttinger Nachrichten (S. 603—606 [2]) in Auszug abgedruckten Briefe an R. FRICKE habe ich (S. 605, Fussnote [3]) kurz skizziert, wie das von HURWITZ herrührende analytische Theorem, dass *birationale Transformationen einer Riemannschen Fläche vom Geschlechte $p > 1$ in sich unmöglich ein vollständiges kanonisches Schnittsystem der Fläche in ein äquivalentes kanonisches Schnittsystem überführen können*, mittels der Analysis Situs bewiesen werden kann, wobei sich seine Gültigkeit herausstellt für *alle periodischen, eineindeutigen und stetigen Transformationen*. Die damalige Andeutung wird im folgenden näher präzisiert und gerechtfertigt werden.

Sei O die gegebene zweiseitige Fläche, I die Menge der bei der n-periodischen, eineindeutigen und stetigen, die Ränder invariant lassenden Transformation t von O invarianten Punkte. Wir nehmen an, dass jedes von I in O bestimmte Gebiet von t in sich transformiert wird (was, wenn t die Indikatrix von O invariant lässt, stets der Fall ist), und unterziehen die Wirkung von t auf eines dieser Gebiete, welches wir mit ω bezeichnen werden, einer näheren Betrachtung. Dabei ziehen wir im Falle, dass t die Indikatrix von O umkehrt, jeden eventuellen für t *nicht* invarianten Rand von ω

[1]) Vgl. diese Proceedings, XI, S. 788; XII, S. 286; XIII, S. 767; XIV, S. 300; XV, S. 352.

[2]) Das daselbst S. 604 auf künftige Publikationen von P. KOEBE (der Neujahr 1912 im Besitze einer Abschrift meines Briefes an R. FRICKE war) hinweisende Zitat ist nach der Erledigung der Korrekturen von einer mir unbekannten Hand, ohne meine Mitwirkung oder Vorkenntnis eingefügt worden; die bezüglichen Noten sind mir erst nach ihrem Erscheinen bekannt geworden.

In engem Zusammenhang mit dem Inhalte meines (Anfang März 1912 gedruckten) Briefes an R. FRICKE stehen die Karlsruher Verhandlungen über automorphe Funktionen vom Jahre 1911; der über dieselben erstattete Bericht (Jahresber. d. D. M. V. XXI), ist, ebenso wie die in Gött. Nachr. 1912 erschienene KOEBE'sche Mitteilung über den Kontinuitätsbeweis, im Sommer 1912 gedruckt worden. Das in diesem Berichte enthaltene Referat über den Vortrag von P. KOEBE (insbesondere die auf S. 162 befindliche Anmerkung [1]) ist insofern irreführend, dass im wirklichen KOEBE'schen Vortrage, nach der ihm vorangegangenen, S. 156—157 wiedergegebenen kurzen Diskussion mit mir, vom Kontinuitätsbeweise überhaupt nicht wieder die Rede gewesen ist.

in einen Punkt zusammen und fügen diesen Punkt zu ω hinzu. Sei R die Menge der übrig bleibenden (für t invarianten) Ränder r_α von ω und seien (a_1, b_1), (a_2, b_2), (a_3, b_3), ... die Rückkehrschnittpaare einer *vollständigen* kanonischen Zerschneidung von ω. Wir nehmen an, dass innerhalb jedes dieser Rückkehrschnittpaare ein a_ν so gewählt werden kann, dass nicht nur a_ν selbst, sondern auch seine beiden Seiten von t ungeachtet der Ränder äquivalent abgebildet werden (was, wenn t jeden Rückkehrschnitt der kanonischen Zerschneidung ungeachtet der Ränder äquivalent abbildet und die Indikatrix von O umkehrt, stets der Fall ist). Weiter schliessen wir den Fall aus, dass entweder kein Rückkehrschnittpaar (a_ν, b_ν) und höchstens zwei Ränder r_α, oder kein Rand r_α und nur ein einziges Rückkehrschnittpaar (a_ν, b_ν) existiert.

§ 2. Im Falle, dass in ω wenigstens zwei Rückkehrschnittpaare (a_ν, b_ν) existieren, verstehen wir unter Ω diejenige (kein Rückkehrschnittpaar mehr aufweisende) Schottkysche Ueberlagerungsfläche von ω, welche die a_ν als blättertrennende Ufer besitzt, unter L die Menge derjenigen Ränder l_α von Ω, welche durch eine unendliche Zahl von Ueberschreitungen der a_ν auf ω erzeugt werden, unter ${}_1a_\nu, {}_2a_\nu, \ldots$ die Ueberlagerungsbilder von a_ν auf Ω, unter a'_ν das durch t auf ω bestimmte Bild von a_ν, unter ${}_1a'_\nu, {}_2a'_\nu, \ldots$ die Ueberlagerungsbilder von a'_ν auf Ω. Wenn wir jedem ${}_\mu a_\nu$ dasjenige ${}_\lambda a'_\nu$ zuordnen, dessen Umlaufkoeffizienten zwischen den l_α die gleichen absoluten Werte besitzen, wie die entsprechenden Umlaufkoeffizienten von ${}_\mu a_\nu$, so ist in Anschluss daran eine durch die Ueberlagerung von Ω über ω in t übergehende, eineindeutige und stetige Transformation t' von Ω in sich bestimmt, welche, ebenso wie t, n-periodisch sein muss und jeden Rand l_α invariant lässt. Ob *alle* Ränder von Ω für t' invariant sind, lassen wir dahingestellt, ziehen aber jeden eventuellen für t' *nicht* invarianten Rand in einen Punkt zusammen und fügen diesen Punkt zu Ω hinzu.

Im Falle, dass in ω nur ein einziges Rückkehrschnittpaar (a, b) existiert, verstehen wir unter Ω diejenige (kein Rückkehrschnittpaar mehr aufweisende) Schottkysche Ueberlagerungsfläche von ω, welche a als blättertrennendes Ufer besitzt, unter l_1 und l_2 diejenigen Ränder von Ω, welche durch eine unendliche Zahl von Ueberschreitungen von a auf ω erzeugt werden, unter ${}_1r, {}_2r, \ldots$ die Ueberlagerungsbilder auf Ω eines (für t der Annahme gemäss invarianten) Randes r von ω. Alsdann existiert eine durch die Ueberlagerung von Ω über ω in t übergehende, eineindeutige und stetige Transformation t' von Ω in sich, welche ${}_1r$ (mithin auch ${}_2r, {}_3r, \ldots$) invariant lässt,

so dass sie, ebenso wie t, n-periodisch sein muss. Ob *alle* Ränder von $\mathfrak{Q}$ für t' invariant sind, lassen wir wieder dahingestellt und ziehen jeden eventuellen für t' *nicht* invarianten Rand in einen Punkt zusammen, den wir zu $\mathfrak{Q}$ hinzufügen.

Im Falle, dass in ω kein Rückkehrschnittpaar (a_ν, b_ν) existiert, verstehen wir unter $\mathfrak{Q}$ die Fläche ω selbst und unter t' die Transformation t selbst.

In jedem der drei Fälle besitzt die Fläche $\mathfrak{Q}$ *wenigstens drei* Ränder und ist sie eineindeutiges stetiges Bild eines Teilgebietes der Kugel, während t' eine n-periodische, eineindeutige und stetige Transformation von $\mathfrak{Q}$ in sich darstellt, welche jeden Rand, aber keinen Punkt invariant lässt.

Wir dürfen annehmen, dass von den Rändern von $\mathfrak{Q}$ *wenigstens einer isoliert* ist. Im entgegengesetzten Falle können wir nämlich in $\mathfrak{Q}$ ein Gebiet g bestimmen, zu dessen Grenze kein Grenzpunkt von Rändern von $\mathfrak{Q}$ gehört, und welches, ebenso wie seine Komplementärmenge, Ränder von $\mathfrak{Q}$ enthält. Die Vereinigung von g und seinen Bildern für $t', t'^2, \ldots . t'^{n-1}$ bildet eine Fläche, welche, ebenso wie $\mathfrak{Q}$, wenigstens drei Ränder besitzt und von t' mit invarianten Rändern und ohne invariante Punkte in sich transformiert wird, überdies aber einen isolierten Rand besitzt.

§ 3. Seien $P_1, P_2, \ldots . P_{n-1}$ die Punkte, in welche ein in $\mathfrak{Q}$ willkürlich gewählter Punkt P von $t', t'^2, \ldots . t'^{n-1}$ der Reihe nach übergeführt wird. Ein P und P_1 verbindender stetiger Kurvenbogen j_P bildet mit seinen von $t', t'^2, \ldots . t'^{n-1}$ bestimmten Bildern eine geschlossene stetige Kurve k_P. Die Menge R' der Ränder r'_α von $\mathfrak{Q}$ lässt sich in solcher Weise in eine endliche Zahl ≥ 3 von voneinander isolierten Teilmengen $R'_1(k_P)$, $R'_2(k_P), \ldots . R'_m(k_P)$ zerlegen, dass der Umlaufkoeffizient von k_P zwischen einem zu $R'_\nu(k_P)$ und einem zu $R'_\lambda(k_P)$ gehörigen Rand modulo n gleich einem durch ν und λ bestimmten, für $\nu = \lambda$ fortfallenden Wert $c_{\nu\lambda}(k_P) \geq 0$ und $< n$ ist. Diese $R'_\nu(k_P)$ behalten ihre Brauchbarkeit und die zugehörigen $c_{\nu\lambda}(k_P)$ ihre Gültigkeit bei, wenn für festes P der Bogen j_P diskontinuierlich geändert wird. Mittels gleichzeitiger stetiger Variierung von P und j_P sieht man weiter ein, dass auch durch Aenderung des Punktes P die Rolle der $R'_\nu(k_P)$ und $c_{\nu\lambda}(k_P)$ nicht gestört wird. Indem wir aber P in hinreichender Nähe von R'_h und j_P in passender Weise wählen, können wir dafür sorgen, dass der Umlaufkoeffizient von k_P zwischen einem zu $R'_\nu(k_P)$ und einem zu $R'_\lambda(k_P)$ gehörigen Rand, für ν und λ beide von h verschieden, fortfällt und

hieraus folgern wir unmittelbar, dass $c_{\nu\lambda}(k_P) = 0$ für jedes ν, jedes λ, jedes P und jedes j_P.

§ 4. Wir bezeichnen einen willkürlich gewählten isolierten Rand von $\mathfrak{Q}$ mit ϱ, die Menge der übrigen Ränder von $\mathfrak{Q}$ mit R'_0 und betrachten die sich um R' aperiodisch herumwindende Ueberlagerungsfläche S von $\mathfrak{Q}$. Diese Fläche S ist eineindeutiges und stetiges Bild der Cartesischen Ebene; ihr Rand R'' enthält einen aus ϱ hervorgegangenen Teil R_1'' und einen aus R'_0 hervorgegangenen Teil R_2''; diese beiden Teile sind voneinander isoliert.

Sei P ein Punkt von $\mathfrak{Q}$ in der Umgebung von ϱ, P_1 sein von t' bestimmtes Bild. Wir verbinden P und P_1 in der Umgebung von ϱ durch einen solchen stetigen Kurvenbogen j_P, welche mit seinen von $t', t'^2, \ldots . t'^{n-1}$ bestimmten Bildern eine geschlossene stetige Kurve k_P erzeugt, deren Umlaufkoeffizient zwischen zwei willkürlichen Rändern von $\mathfrak{Q}$ fortfällt. Die Möglichkeit, einen derartigen Kurvenbogen j_P herzustellen, folgt aus § 3. Sei P_m der Anfangspunkt, P_{1m} der Endpunkt eines Ueberlagerungsbildes von j_P auf S, so existiert eine durch die Ueberlagerung von S über $\mathfrak{Q}$ in t' übergehende, eineindeutige und stetige Transformation t'' von S in sich, welche P_m in P_{1m} überführt. Alsdann lässt die Transformation t''^n den Punkt P_m invariant, so dass t'', ebenso wie t', n-periodisch sein muss. Hiermit sind wir aber zu einem Widerspruch gelangt, weil eine periodische, eineindeutige und stetige Transformation der Cartesischen Ebene in sich ohne invariante Punkte nicht existieren kann.

§ 5. Im Falle, dass t die Indikatrix von O invariant lässt und von einer vollständigen kanonischen Zerschneidung von O jeden Rückkehrschnitt samt seinen beiden Seiten ungeachtet der Ränder äquivalent abbildet, besitzt t dieselbe Eigenschaft in bezug auf ω (was unmittelbar wie folgt eingesehen werden kann: Sei s ein Rückkehrschnitt von ω, der ω nicht zerlegt, so entspricht einer stetigen Variierung von s in O, wenn die Ränder von ω je in einen zu ω hinzuzufügenden Punkt zusammengezogen werden, eine stetige Variierung von s in ω). Wäre nun ω samt seiner Grenze nicht identisch mit O, so besässe ω einen *durch eine zusammenhängende perfekte Menge von für t invarianten Punkten abgeschlossenen Rand* und würde auf Grund davon die in den §§ 3 und 4 hinsichtlich der Transformation t' von $\mathfrak{Q}$ angestellte, auf einen Widerspruch führende Ueberlegung *auch im Falle, dass in ω kein Rückkehrschnittpaar (a_ν, b_ν) und nur zwei Ränder existieren*, in Kraft bleiben. Mithin ist in diesem Falle ω *samt seiner Grenze mit O identisch und O entweder eine Kugel, oder ein Zylinder, oder eine Cartesische Ebene, oder ein Torus.*

Physiology. — *"The Significance of the Size of the Neurone and its Parts."* By Prof. EUG. DUBOIS. (Communicated by Prof. H. ZWAARDEMAKER.)

(Communicated in the meeting of October 26, 1918).

The existence of definite relations of quantity of the neurone and its parts to the weight of the body is no longer open to doubt. [1]) For homoneuric species of mammals (species with the same organization of nervous system), whose body weights are to each other in the ratio of $P:1$, the volume of homologous neurones — as the volume (or the weight) of the brain — varies proportionally to $P^{0.56}$; the volume of their central part, the cell body proportionally to $P^{0.28}$. It may be assumed that the ideal values are $P^{0.55..}$ and $P^{0.277..}$.

These relations can best be verified by a study of the peripheral nerve fiber. As this constitutes by far the greater part of the neurone to which it belongs, also the volume of the homologous peripheral nerve fibers varies pretty accurately proportionally to $P^{0.56}$ for homoneuric species. And given that the length of the nerve fiber, for *perfectly uniform* homoneuric species, must vary proportionally to the longitudinal dimension of the animal, i.e. to $P^{0.33}$, the conclusion follows naturally that both, the area of the section and the length of the homologous peripheral nerves of homoneuric species, which in reality — for physiological reasons — cannot be *perfectly uniform*, varies *about* proportionally to the longitudinal dimension of the animal.

For perfect uniformity, the area of the section would have to increase proportionally to $P^{0.66..}$ and the volume proportionally to P, when the nerve length increases proportionally to $P^{0.33..}$ This now, is physiologically impossible, as may appear in what follows. The available numerical data, considered physiologically, really lead to the conclusion that *the length and the area of the section of a nerve increase uniformly*, i. e. both proportionally to $P^{0.28}$, hence in the same ratio as the cell-body becomes more voluminous. This holds both for the neurones with peripheral nerve fibers, which conduct the influxions centripetally, the sensitive nerve fibers, and for the neurones with peripheral nerve fibers, through which influxions are

[1]) These Proceedings. Vol. XX. (1918), p. 1328—1337.

transmitted from the center outward, the motor nerve fibers chiefly. Thus the volume of homologous neurones increases proportionally to $P^{0.56}$.

Also for the volume of homologous neurones in the brain (certainly to be compared, if not morphologically, yet as functional units with the peripheral neurones), the same proportionality may be assumed as for the neurones with peripheral nerve fibers, as appears from the equal relations of quantity holding for brain-weight.

Direct data concerning the relative areas of the sections of homologous nerve fibers, are, however, at our disposal only to a very limited degree as yet. Most of them refer to the eye, more particularly to nerve ends, where the retinal area marks the total area of the receptive terminations of a very important group of centripetally conducting nerves. Many of these data have been furnished to us by LAPICQUE, in collaboration with LAUGIER and WATERLOT [1]) through measurements of the diameter of the eye in a number of Mammals, Birds, Reptiles, and Amphibians; from these we can derive the relative size of the retina in approximation. Though at first (1908) LAPICQUE, in virtue of these measurements, assumed that the diameters of the eye vary pretty nearly proportionally to the power $^1/_8$ of the body weight, he later on (1910) considered as more accurate proportionality the power $^1/_7$ of the body weight. According to the first estimation of LAPICQUE the retinal area would have to vary about proportionally to $P^{2/8}$ or $P^{0.25}$, on the other hand proportionally to $P^{2/7}$ or $P^{0.2857}$ according to his last estimation. The data as such allow, indeed, only estimation of the general result.

Some examples may suffice here, derived in the first place from the tables of LAPICQUE and his collaborators. For the exponent of relation in question, which I shall denote by r_0 in the formula

$$\left(\frac{P}{p}\right)^{r_0} = \frac{O}{o},$$

and

$$r_0 = \frac{\log O - \log o}{\log P - \log p},$$

where P and p denote the body weights, and O and o the retinal areas (more accurately here the areas of section of the eye-ball proportional to these in approximation) I find the following values.

In comparison of the area of section of the eye, computed from the diameter of the eye, of Equus caballus and Antelope (dorcas?) 0.2643, of Canis lupus and Canis vulpes 0.2668, of Felis pardus and

---

[1]) These Proceedings Vol. XX. (1918), p. 1337, Note 1).

Felis domestica 0.2390, of Mustela putorius and Mustela erminea 0.3004; as a mean of these three pair of Carnivores 0.2687.

Of the Reptiles examined by WATERLOT the gigantic Lizard Varanus niloticus, with a body weight of 7500 grams and a diameter of the eye of 5.8 mm., compared with the small Gecko Hemidactylus Brooki (3 indiv.), which weighs only 4.9 grams and has a diameter of the eye of 4.25 mm. on an average, gives the value 0.2942 for r_0. Comparing the same Varan or Monitor with a Green Lizard, Lacerta viridis (examined by LAPICQUE) of a weight of 16.8 grams and a diameter of the eye of 5.8 mm., I find $r_0 = 0{,}2517$. The mean of these two values is 0.2730.

Among Amphibians I mention (again examined by LAPICQUE) Rana fusca with a body weight of 53.0 grams and a diameter of the eye of 6.6 mm., and Hyla arborea, with a body weight of 4.8 grams and a diameter of 4.6 mm. They give $r_0 = 0.3006$.

Two species of Fishes according to WELCKER [1]), Cyprinus carpio and Gobio fluviatilis (2 indiv.), of resp. 1817.3 grams and 42.2 grams body weight, and 1550 and 238 milligrams weight of the eye (from which here the sectional area is calculated) give $r_0 = 0.3290$.

Records of *retinal* areas of some adult species of aquatic mammals, accurately computed from direct measurements, of which some differ very much in the size of their bodies, may be found in PÜTTER's extensive treatise [2]).

The most homoneuric species, Phoca barbata and Phoca vitulina, whose body lengths were in a ratio of 3 : 1.75 to each other, had retinal areas of 2543 and 1980 square millimeters. Assuming uniformity of these animals, we find, for a ratio of weight of only a little more than 5 : 1, $r_0 = 0.2972$.

Of two Toothed Whales, which are, indeed, not so closely akin, but with much greater difference of weight, Hyperoodon rostratus and Phocaena communis, the ratio of length was 6 : 1, and the areas of the retina 5000 and 1225 square millimeters. Assuming again uniformity, we find for the ratio of the body weights 216 : 1 and then $r_0 = 0.2617$. As however the body of Hyperoodon is somewhat slenderer than Phocaena, the real ratio of weight must have been somewhat smaller; the real value of r_0 was, therefore, somewhat higher.

---

1) H. WELCKER—A. BRANDT, Gewichtswerthe der Körperorgane bei dem Menschen und den Thieren. Archiv für Anthropologie, Bd. 28 (1902), p. 60.

2) A. PÜTTER, Die Augen der Wassersäugetiere. Zoologische Jahrbücher, Abteilung für Anatomie und Ontogenie der Tiere. Jena 1903, p. 167, 174, 198, 209, 239, 243, 272, 280. The calculations of the areas of the retinae, not immediately comparable, were made by A. LERCH.

Comparing the gigantic Balaenoptera physalus, which is likewise somewhat slenderer than Phocaena communis, with the latter, of still more distant relationship, but which it exceeds 5323 times in weight calculated according to the length, we find $r_0 = 0.2610$ for a ratio of the retinal areas of 11500 : 1225 square millimeters. This value, too, would certainly become somewhat larger, if the real weight could have been taken into account, instead of what has been found from the length.

The deviations of the found exponents of the value 0.277.. are not very important if it is considered that:

1. The compared species are not all perfectly homoneuric,
2. also the specimens are not always typical for their species,
3. the retinal area can only be calculated from the diameter of the eye (in one case the weight of the eye) in approximation,
4. in other cases the body weight was not directly determined.

In virtue of these and many other data, considered in the light of their physiological significance, I think I may assume that the area of the retina really varies on an average proportionally to $P^{0.28}$ or more accurately $P^{0.277}$ for homoneuric species of Vertebrates, and that the same proportionality is of general application for the area of the section of the homologous nerve fibers.

About these relations for individuals within a species hardly any direct data are available. Lapicque [1]) states that for Canis familiaris the diameter of the eye only varies from 20 to 23 mm., whereas the body weight varies from 5 to 40 kilograms, i.e. about as the fifteenth power root of this weight. For r_0 we find here 0.1344, which means the individual ocular exponent of relation has decreased in the same ratio (taking the degree of accuracy of these measurements into account), with respect to what is observed between different species, as the encephalic exponent of relation. That the found value lies nearer half 0.28 than half 0.22 may be remarked in passing.

Measurements of the thickness of homologous nerve fibers in individuals of different weight of one species, to form an opinion of the ratio considered here, are entirely wanting, but I think I may deem it probable that the area of the section of the peripheral nerve fibers remains the same, and that on the other hand the white nerve fibers of the brain vary both in length and in section proportionally to $P^{0.14}$ (or $P^{0.11}$?), which is in connection with the absence of the

[1]) L. Lapicque, La grandeur relative de l'oeil et l'appréciation du poids encéphalique. Comptes rendus des séances de l'Académie des Sciences. Paris 1908 (2). Tome 147, p. 210.

nodes of Ranvier at those nerve fibers in the brain, and with the fact that the number of nodes at the peripheral nerve fibers remains the same for large and small individuals of the same species, whereas between different species this number varies proportionally to the length of the peripheral nerve fiber. [1])

The meaning of the term $P^{0.28}$ must be looked for in the physiological function of the nerve fiber and the retina, namely the conduction of the impulsions or influxions. Likewise of the same relation between the body weight and the volume of the ganglion cell, which volume must undoubtedly be in relation with the quantity of the impulsions which it can receive and emit.

GÖTHLIN (in 1907) was the first to pronounce the idea that the rate of conduction of the nerve influxion in an axis cylinder seems to be in a definite relation to the diameter of the axis cylinder and to the thickness of the medullary sheath, when he tried to apply WILLIAM THOMSON's cable formulae to the white nerve fiber [2]). Ten years later GÖTHLIN justly considers the data enabling us to judge about this relation, deplorably few [3]). He reminds (in 1917) of the fact that according to the exceedingly important researches by CHAUVEAU in 1878, the conduction in the motor nerve fibers of the larynx of the Horse is about 8 times more rapid than in the motor nerve fibers of the oesophagus of this animal [4]), and he brings this difference in rate of conduction of the "influx nerveux" in connection with the very different thickness of the nerve fibers in question [5]). In the very thin and at the same time non-medullated fibers of the splenic nerve of the Ox the rate of conduction, according

---

[1]) A. E. BOYCOT, On the Number of Nodes of Ranvier in Different Stages of the Growth of Nerve Fibres in the Frog. Journal of Physiology. Vol. 30 (1904). London, p. 370—380. At this place we find also a comparison of Cavia porcellus and Mus musculus.

[2]) G. F. GÖTHLIN, Experimentella Undersökningar af Ledningens Natur i den Hvita Nervsubstansen. Uppsala 1907, p. 120 seq.

[3]) Relation entre le fonctionnement et la structure des éléments nerveux. Upsala 1917, p. 15.

[4]) A. CHAUVEAU, Vitesse de propagation des excitations dans les nerfs moteurs des muscles rouges de faisceaux striés, soustraits à l'empire de la volonté. Comptes rendus de l'Académie des Sciences. Tome 87. Paris 1878, p. 238—242. There, p. 138—142, also: Vitesse de propagation des excitations dans les nerfs moteurs des muscles de la vie animale, chez les animaux mammifères.

[5]) A. VAN GEHUCHTEN and M. MOLHANT, (Contribution à l'étude anatomique du nerf pneumogastrique chez l'Homme. Le Névraxe, Vol. 13. Louvain 1912, p. 96) for the Rabbit.

to A. Fischer [1]) is only $^1/_{11}$ of that in those slow oesophagus nerves of the Horse. Göthlin adds to this (p. 16) that only by assuming the influxion to be conducted in the same way in the nerve fiber as electricity in a cable, it is to be understood why the velocity of conduction varies according to the dimensions of the fibers. With the thickening of the medullary sheath, which serves as "relative isolator", the capacity of the cable evidently diminishes, and with the enlargement of the area of section of the "conducting" axis cylinder, the resistance of the cable diminishes. In fact then they equally enhance the conductivity of the nerve fiber, and thus it becomes comprehensible that (as Donaldson and Hoke and also others found) the mean area of section of the medullary sheath in all vertebrates remains equal to that of the axis cylinder, which it envelops. However, Göthlin again justly points this out, the nerve fiber should by no means be imagined as an equally passive conductor as e.g. a telegraph wire. On the contrary, many circumstances render it necessary to assume that in all long nerve fibers which are rapid conductors, the influxion is regenerated in some way or other during its conduction, and thus compensates for the losses of energy during the propagation in an ever enlarged space [2]).

Chauveau evidently supposed a relation between the greater or smaller rate of conduction of the "excitations" in the nerves, and the voluntary or involuntary character of the movements they excite. It is pretty firmly established now that we have to think here of more tangible causes.

Our knowledge took an important step forward by Carlson's researches (in 1904 and 1906) [3]). He demonstrated that for Reptiles (Snakes), Amphibians (Frog), Fishes (the Californian Hagfish Bdellostoma), Cephalopods (Octopus, Loligo), Gastropods (Slug Limax, Ariolimax, Sea hare Pleurobranchaea) and Crustaceans (Spider Crab, Lobster, Limulus) there exists proportionality between the rate of propagation of the impulses in the motor nerve and the contraction

---

[1]) A. Fischer, Ein Beitrag zur Kenntnis des Ablaufs der Erregungsvorgänge im marklosen Warmblüternerven. Giessen 1911. cf. Göthlin, p. 15.

[2]) Cf: J. B. Johnston, On the Significance of the Caliber of the Parts of the Neurone in Vertebrates. Journal of Comparative Neurology and Psychology. Vol. 18. Philadelphia 1908, p. 609—618.

[3]) A. J. Carlson, The Rate of the Nervous Impulse in the Spinal Cord and in the Vagus and the Hypoglossal Nerves of the Californian Hagfish (Bdellostoma Dombeyi). American Journal of Physiology. Vol. X. Boston 1904, p. 401—418.

——, Further evidence of the direct relation between the Rate of Conduction in a Motor Nerve and the Rapidity of Contraction in the Muscle. Ibid. Vol. XV. Boston 1906, p. 136—143.

time of the muscle. In every animal "the swifter the action of the muscle, the greater the rate of propagation of the impulse in the motor nerve supplying the muscle". All the nerves are, so to say, tuned to the muscle they supply.

This relation having been established, LAPICQUE and LEGENDRE [1]) examined which anatomic characteristic of the nerve fibers answers to those physiological properties. They found for the common Frog (Rana esculenta) that the thickness of the nerve fibers regularly increases with the rate of conduction of the influxion, to be measured by the rapidity of contraction in the muscle. It may be derived from the values communicated by them that the rapidity in question in the nerve fiber varies in geometrical ratio with the *area of the section* of the nerve fiber. They also found with regard to the Rabbit, that among others the nerve fibers for the rapid adductor magnus muscle are thicker than those for the semitendinosus, which is a slow muscle. These data make it highly probable that the principle holds good universally: "Les fibres nerveuses sont d'autant plus rapides qu'elles sont plus grosses."

LAPICQUE demonstrated further that the movements of different Amphibians are the quicker or the slower as the rate of propagation of influxions in the nerves of the hind-legs is greater or smaller. [2]) For the slow Common Toad (Bufo vulgaris) the rate of conduction in the nerve for the musculus gastrocnemius is only about half so great as for the common Frog (Rana esculenta), which jumps quickly and far. The Frog-toad (Pelobates fuscus), far exceeds the Toads proper in the rapidity of its movements, and has, in agreement with this, equally rapid nerves in its hind-legs as the Frog. But also the Green Toad (Bufo viridis), which is more agile and quicker than the Common Toad, which makes comparatively big jumps, and swims and climbs well, is on a par with the Frog as far as the rate of conduction in the nerves is concerned. Though the Walking Toad (Bufo calamita) does not, indeed, jump like the Frog, it runs almost as fast as a mouse (hence its other name *cursor*), it swims nimbly and rapidly, climbs better than any other Toad

---

[1]) L. LAPICQUE et R. LEGENDRE, Relation entre le diamètre des fibres nerveuses et leur rapidité fonctionnelle. Comptes rendus de l'Académie des Sciences. Paris 1913 (2). Tome 157, p. 1163—1166. Also: La rapidité fonctionnelle des fibres nerveuses mesurée par la chronaxie et son substratum anatomique. Bulletin du Muséum d'histoire naturelle. Année 1914, N°. 4, Paris 1914, p. 248—252.

[2]) LOUIS LAPICQUE, Rapidité nerveuse des membres postérieurs chez divers Batraciens anoures. Bulletin du Muséum d'histoire naturelle, Année 1914, N°. 6, p. 363—366.

and easily digs holes, (casting out the earth with its hind-legs); it has about the same rate of conduction in the nerve as the Frog. The same thing is the case with the Tree-Frog (Hyla arborea), which pounces upon winged insects as its prey.

Now, for a slow American Toad (Bufo lentiginosus), which has the same size as our Common Toad, the mean diameter of the fibers of the nervus ischiadicus is, according to DONALDSON and HOKE, 11.2 micra, as against 14.7 for a frog specimen of about the same weight, belonging to Rana virescens. In a common American Lizard (Sceloporus undulatus) of a body weight of 8.2 grams, the mean diameter of the nerve fibers in the plexus brachialis was 9.8 micra, on the other hand only 6.2 micra in the Horned Toad (Phrynosoma cornutum), which is at least six times heavier, and owes its name to the fact that it moves more like a toad than as the proverbially quick lizard, to the family of which it belongs. For an equally great rapidity as the said Lizard the nerve fiber of the so much heavier Horned Toad would have to double its diameter (to 12.7 micra). The American Turtle Chrysemys marginata, though probably weighing scarcely less than a kilogram, i.e. certainly as much as a hundred times more than the said small Lizard, has nerve fibers of a mean diameter of no more than 12.4 micra, in its plexus brachialis. [1]) It would have to amount to 18.7 micra for equal rapidity as that Lizard.

GÖTHLIN also points out that Shrimps of the genera Crangon, Palaemon and others, which are among the quickest animal species, possess thick nerve fibers provided with medullary sheaths. [2]) L. and M. LAPICQUE found the greatest rapidity of nerves and muscles of all the Invertebrates which they examined, in the tail of Palaemon. [3])

ALCOCK was the first to inquire into the possible influence of the *size of the body* on the rate of propagation of the influxion in nerves, by experiments on the nervus ischiadicus of the Frog, and, externally, in the nervus medianus of Man. [4]) He finds for Man, as

---

[1]) Cf. these diameters of nerve fibers in H.H. DONALDSON and G. W. HOKE, On the Areas of the Axis Cylinder and Medullary Sheath as seen in Cross Sections of the Spinal Nerves in Vertebrates. Journal of Comparative Neurology and Psychology. Vol. 15, Philadelphia 1905, p. 9—11.

[2]) G. F. GÖTHLIN, Die doppelbrechenden Eigenschaften des Nervengewebes. Kungl. Svenska Vetenskapsakademiens Handlingar, Ny Följd, Bd. 51 (1913), p. 84.

[3]) LOUIS et MARCELLE LAPICQUE, Quelques chronaxies chez les Mollusques et Crustacés marins. Comptes rendus de la Société de Biologie. Année 1910. Vol. 2 (69). Paris 1910, p. 280.

[4]) N. H. ALCOCK, On the Rapidity of the Nervous Impulse in Tall and Short Individuals. Proceed. Roy. Society, Vol. 72 (1903). London 1904, p. 414—418.

well as for the Frog, that in all the examined individuals, of one species and in the same nerve, the rate (per unity of length) is the same, hence independent of the size of the body. Small differences in the values of the rate, viz. of on an average 67.5 m. per second for two men of a mean height of 1887 mm. and on an average 65.9 m. for two men of a mean height of 1721 mm., are neglected, evidently considered as experimental errors. The mean body weights of these men may be put at 85 and 67 kg. according to HASSING's tables, [1]) and by means of this an exponent of relation for the velocity of impulsion of 0.1007 can be calculated, to which, however, in itself, not much significance should be ascribed.

Of greater importance, for the study of the influence of the size of the body on the rate of propagation of impulsions in homologous nerves are MÜNNICH's experiments. [2]) The rate (66 m. in the nervus medianus) found by him for Man is in good concordance with the mean rate according to ALCOCK and the latest determination by HELMHOLTZ and BAXT (64.56 m., in 1870), which speaks, indeed, for the reliability of the method. The greatest importance for our subject have, however, the determinations of the rapidity in the nervus ischiadicus of some mammals, viz. three dogs of different sizes and breeds, two cats, and a rabbit. MÜNNICH justly lays stress on the fact that the reliability of the results of his experiments on animals must be greater than those of experiments on Man, where the nerve cannot be laid bare. Besides, it is of importance for a judgment of the influence of the dimension of the nerve on the rate of conduction of the nerve influxions that among MÜNNICH's dogs a large and a small specimen was examined, the body weights of which inter se differed more than six times as much as those of the tall and the short individuals examined by ALCOCK. MÜNNICH found not inconsiderable deviations between a large and a small dog, with which the breed had nothing to do, and a particularly important deviation in a representative of the so remarkable breed of Dachshunds. Also the rapidities found for cats are of great importance. Though MÜNNICH himself, not realizing that the found numerical differences could have any meaning, leaves undecided whether they rest on different rates of conduction or are only caused by the defectiveness of the method, it has now been raised above doubt,

[1]) H. VIERORDT, Anatomische, physiologische und physische Daten und Tabellen. 3 Aufl. 1906, p. 589.

[2]) FERDINAND MÜNNICH, Ueber die Leitungsgeschwindigkeit im motorischen Nerven bei Warmblütern. Zeitschrift für Biologie. Bd. 66. München und Berlin 1916, p. 1—22.

that they have a real meaning. In this connection the much smaller deviation found by ALCOCK between tall and short men, does not even seem to be *entirely* devoid of interest, now that it appears to run parallel with that between the dogs of a much more considerable difference of weight, and these results are in accordance with what was forced on our attention about the significance of the dimensions of the nerves.

For a dog, "brauner Bastard von der Grösse eines Foxterriers," which must have weighed about 7 or 8 kg., MÜNNICH finds a rate of 61 m. per second, equal to that for the Rabbit, which can only have had about one fourth of this body weight. Hence the rapidity for the Rabbit is comparatively great, and we accordingly find for this burrowing Rodent thicker nerve fibers than for the Dog. As mean diameter in the plexus brachialis DONALDSON and HOKE found 13.3 micra for a Rabbit and 11.6 micra for a Shepherd's Dog (probably 8 times heavier). A definite ratio between the rapidity and the area of the section can, however, not be derived from this, because different nerves and different breeds of dogs are compared here.

For his largest cat MÜNNICH found a rate of 81 meters per second, for another specimen, of which he feels it necessary to state that it is adult, and for which he experimented on a shorter length of nerve, which specimen will therefore have been smaller, the rate was 75 meters. As even the largest cat can hardly have reached half the weight of a middle-sized Foxterrier, the rapidity is certainly relatively very large for the Cat. Undoubtedly the considerable thickness of the nerve fibers in the Cat is in connection with this, on an average in the plexus brachialis a diameter of 16 micra, as against 11.6 micra for a dog 5 times heavier according to DONALDSON and HOKE. [1]) Besides, the Cat is also distinguished by particularly large ganglion cells (a peculiarity of all the Felis species). The muscle fibers of the extremities, too, seem to be particularly thick. Already CAVAZZANI had found that the cells of the cervical and of the lumbal ganglia spinalia are particularly large, and equal to the corresponding cells of dogs which are 5 times heavier [2]). Also LEVI found the cells of the fifth cervical ganglion spinale in the cat much larger than in an about equally heavy dog (diameter 81 as against 65.6 micra). They only reached an almost equal diameter (79.7 micra) [3]) in a dog of 7 times the weight (of 23 kg.). According to

[1]) Loc. cit.

[2]) E. CAVAZZANI, Sur les ganglions spinaux. Archives Italiennes de Biologie. Tome 28. Turin 1897, p. 52. Shepherd's Dogs, Pointers, and adult cat.

[3]) G. LEVI, I gangli cerebrospinali. Supplemento al Vol. VII dell' Archivio

LEVI the Cat has also larger ganglion cells in the columnae anteriores of the intumescentia lumbalis of the spinal cord than the Fox, and of all the Mammals examined by him, among which the Ox, the largest pyramid cells in the cerebral cortex [1]). It has appeared, from particularly comparable measurements of HARDESTY, that the mean diameter of homologous cells of the intumescentia cervicalis in the Cat is not much smaller than in the Foxhound, which is 6 times heavier (53.5 micra as against 58.7 micra) [2]). The muscle fibers in the rectus femoris were thicker in the Cat than in any other Mammal that LEVI examined, with the exception only of the Horse; while the mean diameter in the Cat amounted to 55 micra, he found it only 36.2 micra in a dog of about twice the weight (of 6.3 kg.) [3]).

It appears, therefore, very clearly that the quick muscles of the Cat's leg receive rapid impulsions from large ganglion cells through thick nerve fibers.

Something similar holds for the mole-footed Dachshund. Still somewhat more rapidly than in the nervus ischiadicus of a 9 times heavier butcher's dog "von der Grösse einer Deutschen Dogge" did the impulsions propagate in this nerve of a dachshund (at a rate of 88 meters a second as against 85 meters for the butcher's dog).

From the rate of conduction (given by MÜNNICH) of the influxion of 61 meters a second in the nervus ischiadicus of the mongrel dog of the size of a foxterrier and 85 meters a second for the butcher's dog, likewise of a mixed breed, which had the size of a German Boarhound, and must therefore have had about 8 times the weight of the smaller dog of the same species, an interindividual exponent of relation for the rate at which the influxions are conducted in the nervus ischiadicus of 0,1595 can be calculated. This is in satisfactory agreement with the exponent 0.1344 derived above from a record by LAPICQUE for the area of section of the eye-ball within the species Canis familiaris, and the deviations from the value 0.28 are so great, as to justify for both cases the conclusion that here the same relation of the homoneuric species is not valid, but the other relation, the interindividual, which is expressed by an exponent of half the value of that of the species. It does, however, not follow from this that the *area of the section* of the nerve fiber varies in the same ratio as that of the retinal area, for as the

Italiano di Anatomia e di Embriologia. Firenze 1908, p. 177, and: Studî sulla grandezza delle cellule. I. Ibid. Vol. V. Firenze 1906, p. 332.

[1]) Ibid. (1906), p. 334 and 337. Cf. also: K. BRODMANN, Vergleichende Lokalisationslehre der Grosshirnrinde. Leipzig 1909, p. 81.

[2]) Journal of Comp. Neurology. Philadelphia 1902. Vol. 12, N°. 2. p. 160.

[3]) Loc. cit., (1906), p. 327.

length of the nerve fiber varies, from one individual to another within a species, without change of the number of the nodes of RANVIER, where the resistance for the nerve influxion must be much larger than in the internodia, the resistance in the larger animal becomes slighter in inverse ratio to the length of the nerve fiber. I imagine that in this way the nerve fiber, from individual to individual within a species, does not actually become thicker, only longer. This is different for homoneuric species; there the number of internodia, hence also of the nodes, increases in direct ratio to the length of the nerve fiber, which does not bring about a change in the rapidity of the influxion (per unit of length); this is, therefore, directly dependent on the area of the section.

When the propagation of the influxions in the nerve fiber is compared with the motion of electricity in a circuit, in which the resistance (the reciprocal value of the conductivity) is in inverse ratio to the section, and in direct ratio to the length, much becomes clear in the relations of the size in the nervous system that would otherwise remain unaccountable. Then the ganglion cell may, in a certain sense, be compared with an electric condensator or storage battery, which is charged and discharged.

In the first place it becomes clear that only on comparison of homoneuric species the regular quantitative relations of the neurone and its parts to the body weight are found. For the exponent of relation for the volume of the largest ganglion cells of the columnae anteriores in the intumescentia cervicalis the value 0,2387 is found between Horse and Mouse, according to HARDESTY's records, the value 0.3931, on the other hand, between Cat and Mouse; the exponent is only 0.0851 between Horse and Cat. A Cat that had the weight of the Horse, would have a ganglion cell for the motor nerve fiber of the claw muscles 2.763 times as voluminous as the cell belonging to the longest motor nerve fiber of the Horse. HARDESTY points out (p. 166) that the American Hoary Bat (Atalapha cinerea Pal.), which is considerably smaller than the House Mouse, possesses somewhat larger cell bodies in the columnae anteriores of the intumescentia cervicalis than the latter, which he brings in connection with the innervation of the wings. OBERSTEINER also sees a connection between this great functional importance of the forelegs as wings in the Bats and the fact that, while for most Mammals the cells of the intumescentia lumbalis are appreciably larger than those in the intumescentia cervicalis, the reverse is true for Bats [1]).

[1]) H. OBERSTEINER, Bemerkungen zur Bedeutung der wechselnden Grösse von Nervenzellen. (Del Volume Jubilare in onore L. BIANCHI. Catania 1913), p. 4.

From the much older records of KAISER it may be derived that the exponent of relation for the volume of the largest ganglion cells of the cervical medulla of the much larger Bat Plecotus auritus is only 0.1568 in comparison with the Mole [1]). This Bat has, accordingly, a much larger ganglion cell than the Mole, in ratio of equal body weight. OBERSTEINER points out that Man and the Orang-outang have strikingly small cells in their cervical medulla [2]).

According to HARDESTY's measurements homologous cells of the cervical intumescentia of Man are actually somewhat smaller than in the Foxhound, which he exceeds four times in body weight [3]), and not much larger than those of the Cat, which he exceeds more than twenty times in body weight. As well for Macacus, Cynocephalus, Ateles as for Man, the nerve fibers of the plexus brachialis are thin, in comparison with those of other Mammals, the size of the body being taken into consideration.

Evidently the slight quickness of the muscles, but comparative delicacy of the muscle fibers for the hand and the fingers, as necessary factor for their finely regulated collaboration, may be imputed to the slight thickness of the nerve fibers and the comparatively small volume of the ganglion cells, with which they are connected.

The dimensions of the peripheral "conductors" and of the central "accumulators", connected with them, can become of great importance for the quantity of the brain. Where they are to supply muscles which are both strong and quick, the nerve fibers become particularly thick, the cell-bodies large, and thus the neurones voluminous, to which also voluminous brain neurones must answer. Of the problems of the Elephant having a more voluminous brain than the Anthropoid Apes, to which it was thought that an extraordinary mental height had to be attributed for this reason, of the Ateles and Cebus, equally high above the Anthropoids with respect to their quantity of brain, of the aquatic Mammals, among which the Mysticetes possess comparatively larger brain than the Dogs, of the Denticetes, which like the Seals can almost be put on a line with the Anthro-

---

1) O. KAISER, Die Funktionen der Ganglienzellen des Halsmarkes. Haag 1891, p. 63.

2) L. c. p. 4. The same property may be assumed for the Gorilla, whose spinal cord was examined by W. WALDEYER (Das Gorilla-Rückenmark. Abhandlungen der Kön. Preus. Akademie der Wissenschaften Berlin. Jahr 1888. Physikalisch-mathematische Classe. Abt. III, S. 1—147). Cf. HARDESTY, l.c., p. 168.

3) CAVAZZANI (l. c., p. 52 and 53) had already found that the ganglion cells in Man are smaller than in Shepherd's Dogs and Pointers, smaller even than in the Cat.

poids, in this respect [1]), — facts that have led to speculations about uncommonly great intelligence of the Whales — of the comparatively highly cephalised Crocodiles, and of the class of the Fishes, which in general are not inferior to the Reptiles, as far as the quantity of brain is concerned, of all these problems the solution is now obvious [2]). The proboscis of the Elephant, which plays a prominent part in the animal's life, is not only strong and agile, but also provided with very voluminous muscles. That of the Asiatic Elephant measures 2 meters for a length of the body of $3^1/_2$ meters. Quick strong muscles move the long tail of the American Apes mentioned, which muscles are of still greater service to these animals than those of their hands and feet. But also their arms and legs are very long in comparison with the body, especially for Ateles, which owes its popular name of Spider-Monkey to this, and they consequently contain a large mass of muscles. The Whales have an exceedingly quick and strong motor apparatus in their long body for the tail, which admirably like a ship's screw propels the gigantic

---

[1]) Judging from not yet full-grown animals, some investigators have assumed somewhat too great cephalisation. Thus the Seal is not full grown with $12^1/_2$ kg. body weight (LOUIS LAPICQUE, Sur le poids encéphalique des Mammifères amphibies. Bulletin du Muséum d'histoire naturelle. 1912, N°. 1 p. 2). An adult female, examined many years ago by E. H. WEBER (Ueber den Bau des Seehunds, Phoca vitulina. Verhandlungen der Kön. Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Math.-Phys. Classe. Jahrgang 1850, p. 108), however somewhat emaciated in captivity, weighed as much as 43.11 kg. The animal possessed 266.5 grams of brain, from which a coefficient of cephalisation $k = 0.6766$ can be calculated. Of a female Otaria californiana, which had lived in the Amsterdam Zoological Gardens since 1902, and was already fully adult at the time, the body weight was 74 kg., the weight of the brain 374.5 grams, at its death in Nov. 1913, from which $k = 0.7025$ may be calculated. The weight of the body of this animal, too, must, indeed, have been somewhat higher in its healthy state. Accordingly the calculated coefficients of cephalisation are decidedly somewhat higher than normal for both Pinnipeds.

As regards the Whales, we may refer to the weight of a Balaenoptera sulfurea, 22.5 meters long, viz. 63 Am. tons, i. e. 64000 kg. (according to determinations by LUCAS inaccessible to me), which is mentioned in Vol. 12, p. 503 of the fourth edition (1915) of BREHM's Tierleben. The brain, freshly removed, of a Balaenoptera musculus, 18.8 m. (= 60 Norwegian feet) long, weighed 6700 grams, according to G. A. GULDBERG (Forhandlingar i Videnskabs Selskabet i Christiania. Aar 1885. Christiania 1886, p. 128). Assuming uniformity with the other Balaenoptera species, a body weight of 37385 kg. is found for the latter, and in connection with this a cephalisation coefficient of 0.3841.

[2]) The significance of the influence of the organs of hearing, touch and other organs of sense on the quantity of the brain of these Vertebrates is, however, not slight either.

mass with great velocity, and skilfully steers it. The flipper-like hind-limbs of the Seals (Phoca), which form a kind of tail, move the body along through pretty intricate but nimble movements; in other cases (Otaria) very large pectoral fins render such services to the animal, the feet, directed backwards here too, acting more like a helm, worked by powerful muscles. For the Fishes it is again the tail which derives its propelling force from the muscles of the hind part of the body. The Crocodiles have a strong propelling tail. For all these vertebrate aquatic mammals the locomotor muscles must not only be quick, but also particularly voluminous and strong, because of the great density of the medium, in which the movements take place; hence they must consist of thick, but also of numerous muscle fibers. Hence very voluminous and at the same time very numerous neurones. In fact LEGENDRE demonstrated that the nerve fibers of the medulla and the roots of the Dolphin (Delphinus delphis L) are among others much thicker than in Man, the Stag, the Dog, the Rabbit, and the Mouse. He partly accounts for the high cephalisation of this aquatic Mammal by the thickness of its nerve fibers [1]).

Though I do not enter more fully into the discussion of the mutual relation of the dimensions of muscle- and nerve fibers here, I will, however, point out that between homoneuric species the variations of the dimensions of the neurones and their parts, in function of the body weight, *entirely* account for the variations of the weight of the brain, in function of the body weight; hence that the *number* of the neurones (differently from what I thought possible at first) remains the same. This must also apply to the sarco-neurones, hence to the muscle fibers.

That the cell-body of the neurone must become more voluminous, for homoneuric species, in ratio to the arithmetical longitudinal dimension of the body, is a consequence of the already long known dynamic proportion that the weight (the mass) of similar animals augments according to the *cube* of the homologous longitudinal dimensions, the muscular force, on the other hand, proportional to the area of the section of homologous muscles or the *square* of the homologous longitudinal dimensions. Already ninety years ago STRAUS-DÜRCKHEIM [2]) set forth clearly that thus, for similar animals, consequently having the same organization, but of different sizes, the time of every movement must be nearly proportionate to the longitudinal dimen-

[1]) R. LEGENDRE, Notes sur le système nerveux central d'un Dauphin (Delphinus delphis). Bulletin du Muséum d'histoire naturelle, 1912, N°. 1, p. 6—7. Pl. I.

[2]) H. STRAUS-DÜRCKHEIM, Considérations générales sur l'anatomie comparée des animaux articulés. Paris 1828, p. 189 et seq.

sions, although the rate of the locomotion is equal. For this, increase of the accumulated influxion in the cell-body of the neurone is required, and hence of its volume, in the ratio of about $\sqrt[3]{P}$ or $P^{0.33\cdot\cdot}$. In about the same ratio the lengths of the nerve fibers of homoneuric animal species necessarily increase, these lengths being nearly proportionate to the longitudinal dimensions of the animals. At the same time, it may be assumed, as a matter of course, that there is a tendency for the homologous nerve fiber to increase with the least possible deformation. Without any deformation its area of section should increase in the ratio of $P^{0.22\cdot}$, while its length increases in the ratio of $P^{0.33\cdot}$.

On the other hand, the mode of propagation of nervous influxions being similar in homoneuric species of animals, the only possibility to obtain this similarity, i.e. physiological homoneury, under the given conditions, is that *the area of section of the nerve fiber should increase proportionally with its length.* Therefore the area of the section of the fiber has become somewhat greater and its length less than for uniformity of the homologous nerve fibers, both in the ratio of $P^{0.055\cdot\cdot}$. Hence the ratio $P^{0.277\cdot\cdot} : 1$, as was discussed in the outset.

$$P^{0.277\cdot} = P^{0.333\cdot} : P^{0.055\cdot} = P^{0.222\cdot} \times P^{0.055\cdot} = P^{0.277\cdot}.$$

Thus the accumulated influxion, the time and the rate of propagation increase, as they ought to do, in the same ratio.

Through this way of viewing the matter, which starts from the relations of quantity existing in the nervous system, many imponderabilia, such as the mental height, the intelligence, and not seldom also the degree of brain evolution may be eliminated from the foreground of our considerations of the relation between structure and functions of the nervous system, which can only promote the fruitfulness of researches in this region, among others and particularly with regard to the cortex of the brain.

A striking proof for this is furnished by Otto Mayer's researches on the density in which the cells occur in the cortex of the brain of Apes [1]). He namely determined, for seven genera, the mean number of cells per 0.01 cubic millimeter, in 10 fields (according to Brodmann), throughout the whole thickness of the cortex. His results induced him to point out emphatically that the densities by no means run parallel with the order of the examined animals in the zoological system, their degree of brain organization or their intel-

[1]) Otto Mayer, Mikrometrische Untersuchungen über die Zelldichtigkeit der Grosshirnrinde bei den Affen. Journal für Psychologie und Neurologie. Band 19. Leipzig 1912, p. 233—251. 2 Tafeln.

ligence. A result which strongly reminds of the absence of systematical order in the "relative brain weights".

I have, therefore, treated the densities in a similar way as the brain weights, namely by considering them in connection with the weights of the body. For this purpose calculating the mean densities of the whole cortex (over those ten important fields), from MAYER's records, I find a mean density of cell of 1765,4 for the Chimpanzee (Anthropopithecus troglodytes), of 3160 for the Gibbon (Hylobates syndactylus), of 3580.9 for the Capuchin-Monkey (Cebus capucinus), of 3603.4 for the Saimiri or Squirrel-Monkey (Chrysothrix sciurea), and of 3448.1 for the Marmoset (Hapale jacchus), which mean values no more show systematic order than the densities in the separate fields, from which they are computed.

This disorder is replaced by regularity, as soon as the size of the animals is taken into account.

The ratio of the density of cells between the Chimpanzee and the Siamang is as 1 : 1.79. The Chimpanzee has 8 times the weight of this Gibbon-species, and now it is very remarkable that, between these homoneuric species, according to the proportionality demonstrated in my former paper, homologous cells must be more voluminous to a ratio of $8^{0.28} = 1.79$ for the Chimpanzee than for this Gibbon. Between these homoneuric species the density of cells is, therefore, accurately in inverse ratio with the volume of every cell. In other words: for a Gibbon that had the size of a Chimpanzee, the density of cells in the cortex of the cerebrum would be equal to that of the Chimpanzee. It may, therefore, be assumed that these Anthropoids of equal cephalisation (equal quantity of the brain in function of the body weight) are also equal in the organisation of their cortex of the brain. The cells of these two species must be uniform, both as far as the dendrites and the other interstitium is concerned, and with regard to the cell-body.

When Cebus is compared with Hapale, which he exceeds 6 times in body weight (these weights are on an average 1300 and 215 grams; the brain weight of Cebus is 125, that of Hapale is about 8 grams), the cortical density of cells for Cebus is found 1.7 times greater than for a Hapale of equal body weight. In about the same ratio, viz. 1.8 : 1 the cephalisation is, however, also greater for Cebus than for Hapale, a very fair agreement when it is borne in mind that different specimens were compared. Compared with Chrysothrix (Saimiri), (whose body weight amounts to about 400 grams, and which possesses 24 grams of brain) Cebus has 1.5 times higher cephalisation, and a Saimiri of equal body weight as Cebus would

have almost 1.4 times greater density of cells. For these allied American Monkeys the density of cells, in function of the body weight, is, therefore, proportionate to the cephalisation: the greater the quantity of brain, in function of the body weight, the greater the density of cells.

A Cebus, however, of the same body weight as Hylobates syndactylus, would have only two thirds of the density of cells of this long-armed ape, though the latter has three fifths of the weight of the brain of the large Cebus-species in question. Here the density of cells is about in inverse ratio to the cephalisation.

Of special interest is the comparison of the density of the cells of the Chimpanzee with that of Man, both having about the same body weight. HAMMARBERG [1]) determined the density of cells, in normal human brains, through the whole thickness of the cortex, per 0.001 cubic millimeter, i. e. $^1/_{10}$ of MAYER's unit of volume, in different cortical areas, some of which may be compared with the areas according to BRODMANN. Thus in the lobus occipitalis the area striata or field 17 of BRODMANN. Calculated to the same unit of volume as that of MAYER we find here 386 cells, whereas MAYER's chimpanzee has 2888, i. e. 7.5 times as many. Thus 152 in the area gigantopyramidalis or BRODMANN's field 4, as against a density for that chimpanzee of 1172, i. e. 7.7 times as many. In the area frontalis agranularis or field 6 the density for Man is only 111, as against 1136 or 10.2 times as much for the chimpanzee examined by MAYER [2]).

As, according to BRODMANN's [3]) measurements, the entire surface of the cortex in the Chimpanzee is about a third of that in Man, the regio

---

[1]) C. HAMMARBERG, Studien über Klinik und Pathologie der Idiotie, nebst Untersuchungen über die normale Anatomie der Hirnrinde. Upsala 1895.

[2]) In the part of the gyrus frontalis superior, which belongs to the area frontalis agranularis or BRODMANN's sixth field I have *estimated* the density of cells in the deepest half of the third layer. and likewise in the deepest half of the fourth layer of the part of the lobus occipitalis, which corresponds to the area striata or BRODMANN's seventeenth field, from drawings (Table I, Fig. 2 and Table II, Fig. 4), namely according to the ratio of the other densities drawn and also calculated by HAMMARBERG. Thus was also evaluated the density of cells of the insignificant first layer in the part of the gyrus centralis anterior, which belongs to the area gigantopyramidalis or BRODMANN's fourth field.

HAMMARBERG calculated the number of cells per unity of volume from 10 successive sections, each of a thickness of 0.01 mm., or 5 sections of a thickness of 0.02 mm., MAYER on the other hand from only a single section of a thickness of 0.01 mm. As also parts of cells are counted, the number must be slightly greater according to the latter method. Judging by HAMMARBERG's drawings the difference can, however, not be considerable.

[3]) K. BRODMANN, Neue Forschungsergebnisse der Grosshirnrindenanatomie, mit

praecentalis (the fourth and the sixth field) of the Chimpanzee about two thirds of that in Man, the area striata about equal to that of Man, it must be assumed, according to these determinations, that the absolute number of the nerve cells in the cerebral cortex of Man is much smaller than in the Chimpanzee; greatest is the value of the ratio in the area striata (7, against 6 in the regio praecentralis). The human brain being, however, certainly more complicated functionally, (which is generally called highly developed), it would appear that not the number of neurones, but the multiplication of the contiguities of their dendritic processes, which principally constitute the interstitium, corresponds to more complicated (higher) functions.

---

besonderer Berücksichtigung anthropologischer Fragen. Gesellsch. Deutscher Naturforscher und Aerzte. Verhandlungen 1913. Leipzig 1913, p 9, 22 and 25.

The brain weight of the examined chimpanzee amounted only to 295 grams; the average in full-grown state may be estimated at about 400 grams. The ratio of the surface of the cortex to that of Man may, therefore, become somewhat more favourable. With greater brain weight the density of cells in the cortex of the chimpanzee would have become proportionally less, hence the number of cells would have remained the same. The cortex of Man is, indeed, somewhat thicker, in consequence of which the actual values of the ratio must be proportionately smaller than those calculated.

Astronomy. — *"Expansion of a cosmic gassphere, the new stars and the Cepheids"*. By Dr. A. PANNEKOEK. (Communicated by Prof. W. DE SITTER).

(Communicated in the meeting of October 1918).

I.

The new stars which are most fully known in their changes of intensity are of two distinct types. The sudden quick flaming up is common to both; but they differ in their further changes of light. To the one class belong the two brightest Novae of this century, Nova Persei 1901 and Nova Aquilae 1918, as well as Nova Coronae 1866. Immediately after attaining the greatest brilliance the light begins to decrease quickly; then the diminution becomes slower, while a periodicity sets in. In the other class, of which Nova Aurigae 1892 is the best known example — and to which Tycho's star Nova Cassiopeiae 1572 belongs — the star retains its brilliance for a long time, fluctuates irregularly, and finally loses its brilliance rather rapidly. These two types of change of light show a certain correspondence with the two types of light-maximum, long and short, which are observed, alternately in the same star, in the Antalgol stars such as SS Cygni. Whether this analogy is more than an accidental correspondence, or that a real relationship exists, cannot yet be ascertained.

In connection with the appearance of Nova Aurigae SEELIGER has given an explanation which fits the phenomena of this type very well; when a star enters a nebulous mass, thereby being brought to a high temperature, as long as it flies through denser and thinner parts, its temperature will fluctuate up and down. This theory fits the other type less well. Here there is obviously an enormous rise of temperature, caused by a momentary event, of which all the further processes are merely the consequences.

The cause from which this sudden heating arises need not be discussed here. We only put the question of what may be deduced concerning the further events, from simple hypotheses. A cosmic body, suddenly brought to such a high temperature, will not be in equilibrium. It will expand adiabatically, and as a consequence it

will cool down. Usually the loss of heat by radiation is given as the principal cause of the cooling of a star; but the cooling from adiabatic expansion is of much more importance. In a first approximation, therefore, radiation may be neglected. The force of gravity is also left out of account, which to some extent diminishes the force of expansion; this may be done the more legitimately as it is to a greater or less extent compensated by the radiation-pressure. We assume that all changes take place homocentrically.

A volume-element at a distance r_0 from the centre is found by expansion after a time t at a distance $r = r_0 + \Delta$. We must then have the relation

$$\frac{d^2\Delta}{dt^2} = -\frac{1}{\varrho}\frac{dp}{dr}.$$

A volume-element $r_0^2 dr_0 dw$ shifts to the distance r and becomes $r^2\, dr\, dw$. By this the density changes according to:

$$\frac{\varrho}{\varrho_0} = \frac{r_0^2}{r^2 \dfrac{dr}{dr_0}}.$$

As the change takes place adiabatically, $p\varrho^{-1.4}$ remains constant, or $\varrho = \text{Const.} \times p^{5/7}$, therefore:

$$\frac{d^2\Delta}{dt^2} = -\frac{p_0^{5/7}}{\varrho_0} p^{-5/7}\frac{dp}{dr} = -\frac{7}{2}\left(\frac{p_0^{5/7}}{\varrho_0}\right)\frac{dp^{2/7}}{dr}.$$

The index 0 specifies the conditions at the time 0, which thus remain a function of r. We shall indicate by the index 00 the condition for $t = 0$ and $r = 0$, at the centre therefore; putting

$$\left(\frac{p}{p_0}\right)^{2/7} = y, \qquad \frac{p_{00}}{\varrho_{00}} = \alpha, \qquad \frac{p_0}{p_{00}}\frac{\varrho_{00}}{\varrho_0} = \beta;$$

we find

$$\frac{d^2\Delta}{dt^2} = -\frac{7}{2}\frac{p_0}{\varrho_0}\frac{dy}{dr}.$$

Then α is a constant for this gas ball, of the dimension L^2T^{-2}: it is according to $p = \varrho HT$ (H = gas-constant) proportional to the temperature at the centre, and has the physical meaning of the square of the speed of propagation of isothermic disturbances of equilibrium at the centre. β is a number without dimensions, which at the centre is $= 1$, a function of r, which gives the course of α from the centre outwards in the initial condition. The equation of motion now becomes:

$$\frac{d^2\Delta}{dt^2} = -\frac{7}{2}\alpha\beta\frac{dy}{dr}. \quad \ldots\ldots\ldots \quad (1)$$

where y is determined by

$$y = \left(\frac{p}{p_0}\right)^{2/7} = \left(\frac{\varrho}{\varrho_0}\right)^{2/5} = \left(\frac{r_0^2}{r^2\dfrac{dr}{dr_0}}\right)^{2/5}.$$

or

$$y = \left\{\left(\frac{r}{r_0}\right)^2\frac{dr}{dr_0}\right\}^{-2/5} \quad \ldots\ldots\ldots \quad (2)$$

y gives the change of the temperature.

By the formulae (1) and (2) the change of Δ with the time is determined, when the quantity β, which determines the original condition, is known as a function of r. We may, for instance, assume a density distribution such as EMDEN has calculated for a gaseous sphere in equilibrium, but supposing a much higher temperature at each point than belongs to an equilibrium form of this kind. The original conditions must be such that Δ continually increases, owing to the strong force of expansion which is working all the time towards the outside in consequence of the high temperature. This will cause the temperature of each layer to fall in a ratio which is given by the quantity y, and in consequence the luminosity will decrease. The most external coldest layers, which absorb the light of the central parts, will move towards us with great rapidity; *this explains why in all new stars, as soon as the light begins to decrease, the dark absorption-lines are displaced strongly towards the violet* — a phenomenon which it has been attempted in vain to explain by a rapid approach of the whole star, or by differences of pressure.

Even when the initial conditions are simple, the equations (1) and (2) are difficult to integrate. An attempt to find the course of the change by mechanical quadrature failed through the fact that small variations in y come out greatly increased in $\frac{dy}{dr}$, and therefore also in the Δ that is found and the subsequent values for y, so that each step gives an increasing inaccuracy, which, after integration through a few units of time, makes the results quite unreliable. On this account we have not succeeded in explaining the periodic variation in brightness — which both in Nova Persei and Nova Aquilae began to appear after the star had decreased 4 classes of magnitude — by special initial conditions.

On the other hand the general mean course of the process may be calculated. The question may be asked: is it possible for the

whole mass of the star to expand completely uniformly and what are the initial conditions required in this case?

We must then have $r = r_0 f(t)$, where f is independent of r. It follows that

$$\frac{\varrho}{\varrho_0} = f^{-3} \quad ; \quad \frac{p}{p_0} = \left(\frac{\varrho}{\varrho_0}\right)^{7/5} = f^{-21/5} \quad ; \quad \frac{T}{T_0} = \left(\frac{\varrho}{\varrho_0}\right)^{2/5} = f^{-6/5}$$

$$\frac{1}{\varrho}\frac{dp}{dr} = \frac{1}{\varrho_0}\frac{dp_0}{dr_0} f^{-11/5} \qquad \frac{d^2r}{dt^2} = r_0 \frac{d^2f}{dt^2}$$

hence the equation of motion (1) becomes

$$r_0 f'' = -\frac{1}{\varrho_0}\frac{dp_0}{dr_0} f^{-11/5}$$

For $t = 0$ this gives

$$r_0 f_0'' = -\frac{1}{\varrho_0}\frac{dp_0}{dr_0}$$

The quantity f_0'' must be a constant; a special distribution of density and pressure along the radius for $t = 0$ must therefore be sought, in which

$$-\frac{1}{r_0 \varrho_0}\frac{dp_0}{dr_0} = A^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(The dimension of A is T^{-1}); f is then determined by

$$f'' = A^2 f^{-11/5} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

By integration of this equation f is found as a function of the time

$$\tfrac{1}{2}\left(\frac{df}{dt}\right)^2 = -\frac{5}{6} A^2 f^{-6/5} + Const.$$

The constant is determined by the condition that for $t = 0$ and $f = 1$, when the expansion begins, its velocity is 0, hence

$$\left(\frac{df}{dt}\right)^2 = \frac{5}{3} A^2 (1 - f^{-6/5})$$

$$\frac{df}{dt} = A \sqrt{\frac{5}{3}} \sqrt{1 - f^{-6/5}}$$

$$A\, dt = \sqrt{\frac{3}{5}} \frac{df}{\sqrt{1 - f^{-6/5}}}$$

This form can be integrated by partial integration in the form of a series. Putting

$$f^{-6/5} = x.$$

we have

$$\int x^{-11/6}(1-x)^{-1/2} dx = \frac{6}{5} x^{-5/6}(1-x)^{1/2} \left\{ 1 - \frac{2}{1}x - \frac{2.4}{1.7}x^2 - \frac{2.4.10}{1.7.13}x^3 - \ldots \right\} + const.$$

or

$$= 2x^{-5/6}(1-x)^{1/2}\left\{1-\frac{2}{9}(1-x)-\frac{2.4}{9.15}(1-x)^2-\frac{2.4.10}{9.15.21}(1-x)^3-\ldots\right\}+\text{const.}$$

whence

$$At=\sqrt{\tfrac{3}{5}}f\sqrt{1-f^{-6/5}}\left\{1-\frac{2}{1}x-\frac{2.4}{1.7}x^2-\frac{2.4.10}{1.7.13}x^3-\ldots\right\}+1.881$$

or

$$=\sqrt{\tfrac{5}{3}}f\sqrt{1-f^{-6/5}}\left\{1-\frac{2}{9}(1-x)-\frac{2.4}{9.15}(1-x)^2-\frac{2.4.10}{9.15.21}(1-x)^3-\ldots\right\} \qquad (5)$$

The former series may be used when x is small, or f very large, the latter holds for small f, x being near unity. The additive constant in the second series must be equal to 0, since for $t=0$ x and f are both equal to 1. For the first series the constant cannot be determined by this condition: it was found by computing the value of At for one and the same value of x from both series.

By means of these series the time-function At was calculated for a number of values of the expansion-factor f. The temperature is connected with f by the relation

$$T=T_0 f^{-6/5}$$

Since the temperature changes according to this law throughout the entire star, we are entitled to assume that the same law holds for the effective temperature; the radiation per unit area then changes proportionally to T^4, i. e. to $f^{-24/5}$. The surface itself changes as f^2, and thus the luminosity as $f^{-14/5}$, hence

$$\log L-\log L_0=-{}^{14}/_{5}\log f.$$

or expressed in terms of classes of magnitude

$$m-m_0=7\log f. \qquad (6)$$

The following table contains for a number of values of f the corresponding At and $m-m_0$.

f	At	$m-m_0$
1.0	0.0	0.0
1.1	0.455	0.290
1.2	0.655	0.554
1.3	0.815	0.798
1.4	0.956	1.023
1.5	1.086	1.233
1.6	1.207	1.429
1.7	1.322	1.613
1.8	1.433	1.787
1.9	1.541	1.951
2.0	1.645	2.107

t	At	$m - m_0$
2.5	2.137	2.786
	2.135	
3.	2.599	3.340
4.	3.479	4.214
5.	4.327	4.893
10	8.397	7.000
20	16.307	9.107[1])

The intensity thus diminishes slowly at first and then faster and faster, but the velocity soon reaches a maximum, when the star has fallen rather more than 1 magnitude below the original intensity. The velocity of decrease then becomes slower once more and finally approaches a logarithmic curve.

The slow decrease in the beginning is not observed in the new stars, as the process of blazing up has not yet worked out then. Both Nova Persei and Nova Aquilae had their maximum one day after they had reached the first magnitude, and Nova Persei one day before that had already attained the 3rd magnitude. As the starting-point $t = 0$ we must not therefore take the moment of maximum luminosity, but one or two days earlier. Then follows a rapid decrease which, however, soon becomes slower and is then accompanied by periodical variations. On comparing the observed light-curve and the one here calculated they are found not to agree during the further course of the change; the mean observed intensity decreases much more slowly than according to the above calculation. Evidently other influences are at work here, lying outside the simple theory here given. It is therefore only for the first period of rapid decrease of luminosity that agreement may be looked for.

For Nova Persei we shall take 0,0 as the ideal maximum intensity, a little higher than the greatest brightness observed, because for it the final stage of the blazing up overlaps the beginning of the expansion, and for the ideal starting-point the 21st of February. The following values of At are then found from the values of $m - m_0$ on the smoothed observational light-curve.

Date	m obs.	At	t	quotient	A
Febr. 25	1.00	0,97	4^d	0.24	$2{,}8.10^{-6}$
„ 27	1.62	1.35	6	0.22	2,6 „
Mrch 1	2.07	1.62	8	0.20	2,3 „

[1]) From 1.0 to 2.5 equation 5*b* has been used, from 2.5 to 20 equation 5*a*.

Date	m obs.	At	t	quotient	A
Mrch 3	2.42	1.85	10	0.18	$2{,}2.10^{-6}$
„ 5	2.73	2.07	12	0.17	2,0 „
„ 7	3.02	2.30	14	0.16	1,9 „
„ 9	3.27	2.53	16	0.16	1,8 „
„ 11	3.48	2.71	18	0.15	1,7 „
„ 13	3.65	2.88	20	0.14	1,7 „

The diminution of the quotient shows that those influences which later on retard the decrease to a higher degree than the theory requires, begin to manifest themselves even in the first stage. However that may be, the order of magnitude of A as found here cannot but be correct, and from it conclusions may be drawn as to the constitution of the Novae.

When all quantities are expressed in the absolute system, t is measured in seconds; taking 0.21 as a mean value of the quotient in the above table we have

$$A = 0{,}21 : 86400 = 2{,}5 \,.\, 10^{-6}.$$

For Nova Aquilae about the same value is found.

II.

The distribution of pressure and temperature for $t = 0$, which is required for a uniform expansion, and the dependence of A on this distribution are determined by equation (3) in which we shall now leave out the indices 0:

$$-\frac{1}{r\varrho}\frac{dp}{dr} = A^2$$

or

$$\frac{dp}{dr} = -A^2 r\varrho$$

or

$$p = +A^2 \int_R^r \varrho r \, dr \,.$$

where R is the radius of the external surface, where $p = 0$. If a definite density-distribution is assumed as existing at the moment of flaring up, the latter equation determines the pressure as a function of r, and therefore also the temperature:

$$T = \frac{A^2}{H\varrho} \int_R^r \varrho r \, dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

For the density the values have been assumed which EMDEN

has calculated for the equilibrium-forms of spherical cosmic gaseous masses (for $n = 2^1/_2$, $k = 1.4$); the integration has been performed by mechanical quadrature. The integration-intervals were taken four times smaller than the unit of r_1 as used by EMDEN; expressed in our unit the radius of the external surface is 21.67. The result of the integration was as follows:

r	$\log \varrho$	$21.67^2 \int_R^r \frac{\varrho}{\varrho_0} \frac{r\,dr}{R^2}$	$I = \frac{21.67^2\,\varrho_0}{\varrho} \int_R^r \frac{\varrho}{\varrho_0} \frac{r\,dr}{R^2}$
0	0.00000	22,5274	22.53
2	9.95523	20,6746	22.92
4	9.82604	16.0941	24.02
6	9.62544	10.8138	25.62
8	9.36944	6.3898	27.30
10	9.06954	3.3527	28.56
12	8.72945	1.5525	28.94
14	8.34375	0.6148	26.61
16	7.88832	0.1922	24.84
18	7.29284	0.03928	20.04
20	6.32566	0.00315	15.12

The integral I is proportional to the temperature. The result therefore shows, that *the uniform expansion requires a distribution of temperature which differs very little from an even temperature throughout the mass.* If the original process is not a rise of temperature at the surface by friction in a nebulous mass, but if through some catastrophe the entire mass becomes hot throughout, an approximately equal temperature through the whole mass might be expected and in that case, as was here shown, an approximately uniform expansion would take place.

Now for

$$I = \frac{21.67\,\varrho_0}{\varrho} \int_R^r \frac{\varrho}{\varrho_0} \frac{r\,dr}{R^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

we have

$$T = \frac{IR^3A^2}{21{,}67^2.H} = \frac{IR^2A^2\mu}{21{,}67^2.8{,}3.10^7}$$

if μ is the molecular weight of the gas of which the star consists. Substituting the value of A found above, the mean temperature (taking $I = 25$) becomes:

$$T = \frac{25 \times 6.25.10^{-12}}{21{,}67^2 \times 8{,}3.10^7} R^2\mu = 4.10^{-21} R^2\mu \quad . \quad . \quad . \quad . \quad (9)$$

With $T = 10^4$ degrees this gives:

$R^3\mu = 2{,}5.10^{34}$ hence for	$\mu = 1$ (dissoc. H)	$\mu = 50$ (metals)
	$R = 1{,}6 \times 10^{12}$	$2{,}3 \times 10^{11}$
	$= 23$ times the sun	3,5 times the sun

and with $T = 10^5$ degrees

$R^3\mu = 2{,}5.10^{35}$ hence	$R = 5 \times 10^{12}$	7×10^{11}
	$= 71$ times the sun	10 times the sun

Considering that at this high degree of heat the mass will be highly dissociated, the first values are probably nearer the truth than those corresponding to $\mu = 50$. It shows that *a Nova at the moment of greatest brightness is a body much more gigantic than the sun, not only in luminosity but also in radius and volume.* The theory, that a new star arises when a dark body of the size of our sun, i.e. an ordinary cooled-down dwarf star, suddenly rises to a colossal temperature, is in contradiction with the above calculations; for $R = 7 \times 10^{10}$, the radius of our sun, with $\mu = 50$ and A as observed, the temperature would only rise to 1000°; for $T = 10^5$ A would be 10 times larger, that is: the time in which the star loses its light would be 10 times smaller.

This result is in accordance with the value of 0".011 for the parallax of Nova Persei, derived by KAPTEYN from the supposition that the nebulous rings which were photographed half a year later arose from reflected star-light. This leads to a luminosity 10000 times that of the sun; since the intensity of the surface-radiation was not much different from what it is in an ordinary white star — HERTZSPRUNG found a similar distribution of light in the spectrum of Nova Aquilae as in α Aquilae [1]) — the radius of the Nova must have been 30 to 50 times the radius of the sun.

Supposing our interpretation of the dark lines which always accompany the bright lines on the violet side being correct, this also leads us to a high value of R. The velocity with which the outermost particles move towards us is $R\,{}^{df}/_{dt}$. At the moment when the light has fallen by two magnitudes, we have ${}^{df}/_{A\,dt} = 1$, hence $R\,{}^{df}/_{dt} = AR = 2.5 \times 10^{-6}\,R$. For $R =$ the radius of the sun this would become 1.7 km. per second. On the other hand the observed displacement of the dark lines was as much as would correspond to 700 km. per second. The real velocity must have been smaller, however, since the absorption-line is partly effaced by the broad adjoining emission-line; on the assumption that the velocity may have been about 100 or 200 km./sec. R is found equal to 60 or

[1]) Astonomische Nachrichten Bd. 207. Nr. 4950.

120 times the radius of the sun, therefore again a value of the same order of magnitude.

The Novae in the first stage of their brightness thus possess the characteristics of the giant-stars; in order that their mass may not become too exceptionally large, their density must be small even before the expansion. The relation found here between T, R, and A cannot teach us anything on this point, as it does not contain the density. A further indication for a small density may be found, however, in the fact that after a decrease of 4 magnitudes the spectrum at the minima of the light-variations more and more approached the character of a nebula-spectrum, and after another few months the star had become a nebula. At this stage the density has become so small that the visible emission is derived from the whole body including even the hindmost layers and still gives but a feeble surface-brightness; the fact that this condition sets in, when the expansion factor has become something like 10 or 20, proves that the original density must also have been far below unity.

III.

The original equation of motion (1) may also be written in such a form that it does not contain any dimensions.

Let us put

$$r = \gamma s \qquad \Delta = \gamma x \qquad t = \delta z \quad . \quad . \quad . \quad . \quad (10)$$

where γ is a linear measure, δ a length of time and s, x, and z are numerical values. The equations then become

$$\left.\begin{aligned} \frac{d^2x}{dz^2} &= -\frac{7}{2}\alpha\beta\frac{\delta^2}{\gamma^2}\frac{dy}{ds} = -B\frac{dy}{ds} \\ s &= s_0 + x\,; \qquad y = \left(\left(\frac{s}{s_0}\right)^2\frac{ds}{ds_0}\right)^{2/5} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

where β is a function of the coördinate s_0. The function β and the constants α, γ, δ which determine the special constitution and size of the star are united in the one coëfficient B. The law of change of x with z is solely dependent on this coëfficient, and is the same for all bodies with the same B. Equations (11) determine all possible movements — progressive, irregular or periodical, which may occur in a cosmic gaseous mass, in so far as they are a function of r only and as gravitation may be left out of account. Without calculating these movements themselves, a relation of similarity may be derived from the formulae which establishes a connection between the changes in different stars. If for different

cosmic gas-spheres the distribution of β along the radius is the same, the expression

$$x = \frac{\Delta}{\gamma} = f\left(\frac{r}{\gamma}, \frac{t}{\delta}\right)$$

must be the same function for them, provided B i. e. $\alpha\beta\frac{\delta^2}{\gamma^2}$ for them is the same number. If for each of them a suitable time- and distance-scale is assumed, the motions and variations expressed on this scale are for all these bodies identical.

Assuming that a periodical solution of the equations (11) exists in which the particles move radially to and fro and the density periodically becomes adiabatically larger and smaller, this condition of motion will be valid for all such bodies provided the periods of the variations are expressed in δ as unit and the dimensions of the bodies in terms of γ. We must then have the relation

$$\alpha_1 \delta_1^2 \gamma_1^{-2} = \alpha_2 \delta_2^2 \gamma_2^{-2}$$

Now $\alpha = HT_{00}$ (at the centre), therefore proportional to the temperature at the centre. Calling P the period of the variations and R the radius of the gas-sphere, this gives:

$$P_1^2 : P_2^2 = \frac{R_1^3}{T_1} : \frac{R_2^3}{T_2}.$$

If we may assume, that similar bodies of this kind have the same temperature, the brightness becomes proportional to R^2, i. e. to P^2. Otherwise the temperature will still depend on some power of R and we have the more general relation

$$P^2 \backsim L^n$$

or

$$2 \log P = Const. + n \log L$$

or

$$2 \log P = Const. - 0{,}4\, n \times M$$

if M represents the absolute magnitude. A relation of that kind was found by Miss LEAVITT for the variable stars of the δ Cephei-type in the small Magelhanic-cloud [1]). For 25 stars with periods from 1.25 to 127 days she found, that the period increased with the magnitude in such a manner that the logarithm of the period changed by 0.48 per magnitude-class.

The Cepheids are giant-stars, to which our suppositions are in so far applicable, that gravity, small in itself by the small density, must moreover for the greater part be neutralized by the radiation-

) Harvard Circular Nr. 173.

pressure. They are all nearly of the same spectral type, hence their temperature cannot differ much. The relation which has been found to hold for them between period and intensity may therefore be explained in a simple manner by assuming that the variation of light arises from a pulsation of the gaseous sphere; not, as is often assumed, a pulsating deformation, but a pulsating expansion and contraction. Hereby the absorbing layers at the front of the star will alternately move away from us and towards us, hence in the spectrum a periodical displacement will take place. This displacement has usually been taken as indicating an orbital movement and for this reason the Cepheids are admitted amongst the spectroscopic double stars. Still amongst these they occupy a very exceptional position. Calculating the mass from the elements of the orbit, very much smaller values are found for the Cepheids than for other spectroscopic double stars, although their volume is much larger than that of the sun. Although an extremely small density is not altogether impossible a priori, still in the relatively small radial velocity an indication may be seen for the assumption, that a different explanation must be given here than for ordinary spectroscopic double stars.

But the question arises: is it possible that from an expansion and contraction a radial velocity arises of such a value as the experiments give — of several times ten kilometers per second?

The luminosity of δ Cephei and η Aquilae was found by ADAMS from the spectrum to be 60 times that of the sun; for a mean Cepheid with a period of 6.6 days HERTZSPRUNG derived from the proper motions 600 times the luminosity of the sun. Assuming on the ground of the accordance as to spectral type and colour an equal radiating power per unit surface, these results give a radius equal to 8 and 24 times respectively that of the sun. Representing the maximum expansion and contraction by the factor $f = 1 + \Delta f$, the maximum radial velocity will be

$$V = \frac{2\pi \Delta f . R}{86400 P},$$

where P is the period in days and R the radius. In kilometres R is 8 or $24 \times 7 \times 10^5$. Taking for P 6 days, this gives

$$V = \frac{6{,}3 \Delta f \times 8 \text{ (of } 24) \times 7.10^5}{4{,}3 \times 10^5} = 82 \text{ resp. } 246 \Delta f \text{ KM.}$$

Since these Cepheids fluctuate rather less than 1 magnitude visually and rather over 1 magnitude in photographic intensity, we shall assume one magnitude for the variation in complete radiation;

therefore *log* L varies by the amount 0.20 above and below the mean. If the radius changes as the number f, the density changes as f^3, the temperature as $f^{6}/_{5}$ and the radiation as $f^{24}/_{5}$; from *log* $L = \pm 0.20$ it then follows that *log* $f = \pm 0.04$, hence f fluctuates between 1,1 and 0,9. In the expression for V we must therefore take 0,1 for Δf and the maximum radial velocity becomes 8 or 25 kilometers per second. This value has to be somewhat lowered, since spectographically the mean velocity of the entire front surface is measured, of which only the central parts have the velocity which we have here calculated. But even then the value found agrees sufficiently with the measured velocities (10 to 20 kilometers per second) to admit the explanation of the light variation and the variation in radial velocity on the ground of contraction and expansion.

There are some other objections to this explanation. The one is the same objection which also holds against the explanation through an orbital movement viz. that the maximum intensity coïncides with the highest velocity towards us. The other objection lies in the coefficient 0.48 found by Miss Leavitt. If for these Cepheids equality of spectral class and thus of emissive power and of T may be assumed, the brightness becomes proportional to the surface, which gives

$$P^2 \backsim L.$$

or

$$log\ P = Const. - 0{,}2\ M.$$

In this case therefore the coëfficient should be 0.2, whereas Miss Leavitt finds a much larger change of the period or a much smaller change of the brightness. It is therefore difficult to explain the deviation by means of a dependence of the temperature T on the linear dimension R; for in that case T would have to be smaller, the larger the star. Possibly an explanation may be found by assuming, that the mass of the Cepheids is actually small, and therefore the density very low, so low, that the rays emitted from one side of the star may penetrate the complete body without being completely absorbed. If a glowing gas-sphere is so rare, that we observe the emission even from the hindmost layers without any diminution, the total light from the sphere will no longer be proportional to its surface, but to its mass, therefore be the same for two bodies of equal mass and different dimension. Intermediate conditions are conceivable in which the total light will then be proportional to a lower power of R, say to the first power. In the latter case the coëffficient of M in the formula for *log* P would become about 0.40.

Physics. — *"On the Theory of the Friction of Liquids"*. By Prof. J. D. VAN DER WAALS JR. (Communicated by Prof. J. D. VAN DER WAALS).

(Communicated in the meeting of November 30, 1918).

§ 1. *Introduction.* The theory of the friction of gases has been made the subject of numerous researches, the theory of friction of liquids on the other hand has met with but scant attention. Yet it is clear that the explanation given to account for the friction of gases — viz. that it is brought about in consequence of this that molecules diffusing from one gas layer to another, at the same time transport an amount of momentum from one layer to another — cannot equally apply to the friction of liquids. For the friction of gases increases at higher temperature. For liquids on the other hand the viscosity becomes slighter at higher temperature. Such a behaviour cannot be accounted for with "friction by means of transport."

Maxwell calculated that on the supposition of "friction by means of transport" the coefficient of friction η should be proportional to $\sqrt{T}$ if we assume that the molecules are perfectly rigid spheres, which do not attract each other. Other assumptions concerning the nature of the molecules (repulsion in inverse ratio with the fifth power of the distance, MAXWELL, or mutually attracting rigid spheres, SUTHERLAND and REINGANUM) lead to a still more rapid increase of η with T. Nor can the thermal expansion of the liquids explain the sign of the coefficient of temperature of η. For gases η appears to be independent of the volume. For liquids the expansion will promote an increase of η with T, and not a decrease. This has been shown experimentally (except for water, where the reverse takes place), and it is also easy to understand that this is to be expected for friction by means of transport, at least for not associating or dissociating liquids. The expression derived by MAXWELL:

$$\eta = \tfrac{1}{3}\varrho ls = \tfrac{1}{3}\frac{ms}{\pi\sigma^2\sqrt{2}} \quad \ldots\ldots\ldots \quad (1)$$

in which ϱ represents the density, l the mean length of path, s the mean velocity, m the mass, and σ the diameter of a molecule, will namely have to be corrected for liquids, to:

$$\eta = \tfrac{1}{3}\frac{ms}{\pi\sigma^2\sqrt{2}}\frac{v-b}{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

which quantity increases with v. Other well-known corrections have been left out of consideration.

Accordingly for liquids we shall not principally have to think of transport of momentum by the diffusing molecules, but we shall have to explain the friction by forces which the molecules exert on each other. If at an arbitrary moment we could suddenly check the motion of the molecules, and if we could arrest them in the position which they occupied at that moment, the friction by means of transport would at the same moment be destroyed, so that we should not have any means to study the friction in those resting molecules.

The case is different for "friction through molecular forces". At least when we think the molecular forces independent of the velocity, the frictional forces would continue to exist also after the immobilisation of the molecules. They would be a consequence of the grouping of the molecules in space. It is now the question: of what nature are the molecular forces and what is the grouping of the molecules, which gives rise to the existence of the tensor of tension as we meet with it for the friction of liquids. The following three answers might be given to this question:

I. *Friction through impact forces or through an instantaneous transfer of momentum.* We might assume that the forces that the molecules exert on each other at an impact would furnish the explanation of viscosity. Let us consider the simple case of a liquid in which the current only moves in a single direction, which has been chosen as x-direction of a cartesian system of coordinates, this velocity (u) being a linear function of z, hence:

$$u = az, \text{ (with } a > 0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then the layers with greater z will move towards the righthand side with regard to the underlying layers, if the system of axes is orientated in space in the usual way. A consequence will be that the line connecting the centres of two colliding molecules, which I shall call the central line, will be found more often in the second quadrant of the xz-surface than in the first. When the system of coordinates is turned over an angle of 45 °/₀, so that the $+x$-axis moves towards the $+z$-axis, and when the new axes are called x' and z', the pressure that the molecules exert on each other will be greater in the z' direction than in the x' direction. It is evident that this agrees with the value of the tensor of tension in this case.

Instead, however, of the calculation of the forces appearing in case of collision, the friction through this cause can also be calculated by means of the momentum that at impact is momentaneously conveyed from the centre of one of the colliding molecules to that of the other. This method of calculation seems simpler and will be carried out in § 2.

II. *Friction for double points. Formation of streaks.* We might also assume the molecules to be electrical or magnetical double points. When they were orientated with regard to each other quite arbitrarily, they would equally frequently repel as attract each other, so that the mean force would be zero. Through the couples which they exert on each other, they will, however, turn so that attraction prevails. When we now assume that molecules that approach each other, are still little orientated, whereas this is the case to a higher degree with molecules that have moved past each other, and recede again from each other, the molecules whose central line lies in the x'-direction will be more orientated on an average than those for which it lies in the z' direction, so that a traction in the x' direction will result, greater than in the z'-direction, which can again account for the tensor of tension.

When we consider more than two molecules whose centres lie on the same line in the x-direction, the couples they exert on each other, will strengthen each other, which can give rise to the formation of a kind of streaks, which still more promotes the friction.

It is difficult to compute the accurate amount of this orientation of the molecule axes; it will be different according as one thinks the rotations of the molecules determined by classical mechanics or by the laws of the theory of quanta. Besides there is no occasion in the experimental data to assume that this case actually presents itself. I shall, therefore, not attempt to calculate the friction according to this hypothesis, though possibly it plays a decisive part in the friction of exceedingly viscous liquids, which present themselves as bi-refringent in case of friction, as likewise in the glassy state.

III. *Friction in consequence of formation of groups.* Finally we can assume the molecules to combine to groups in consequence of their mutual attraction. In liquids at rest these groups will possess spherical symmetry on an average. When, however, a liquid is in a motion for which $u = az$, these spherical groups will be elongated to ellipsoids. This variation of shape will now again give rise to a greater traction in the x'-direction. This cause of friction will probably chiefly make itself felt in the neighbourhood of the critical point. In § 4 and following paragraphs I will make an attempt

to calculate the amount of the friction which is to be ascribed to this cause.

§ 2. *Friction in consequence of impact forces.* For an accurate calculation of the friction through this cause the accurate knowledge of the distribution of the velocities would be required. I shall, however, confine myself here to an approximate method of calculation of about the same nature as the method of calculation of the "friction by means of transport" for gases by MAXWELL in his papers in the Phil. Mag. in 1860. I shall, namely, assume that the distribution of the velocities of the molecules the centres of which lie in a definite layer $z = z_1$ is found by compounding the velocity of the current of the liquid in that layer with a thermal motion for which the unmodified partition of velocities of MAXWELL is thought to hold.

The error that we make on this supposition will probably be smaller for liquids than for gases. The free length of path is namely very small here, and the supposition departs little from MAXWELL's supposition that the molecules have the velocity of current of the layer in which they have collided last. Even when JEANS' correction is taken into account for the persistence of the velocities, we shall have to assign a velocity to the molecules corresponding with the velocity of current of a layer which is only a small fraction of σ removed from the layer in which their centre is situated. I shall disregard this small fraction.

When we now consider a definite horizontal layer, for which we choose $z = 0$, an instantaneous transfer of momentum through this layer takes place at every collison for which the centres of the colliding molecules lie on different sides of this layer. At every impact an instantaneous transfer from above downwards takes place and one in opposite direction. These two quantities are equal and of opposite signs. Hence we may also take into account double the amount of the transfer from above downwards. We shall now first consider the collisions for which the centre of molecule I lies between the planes $z = z_1$ and $z = z_1 + dz_1$ $(0 > z_1 > -\sigma \cos\gamma)$, the central line[1]) forming an angle between γ and $\gamma + d\gamma$ with the z-axis, and lying in a plane forming an angle between β and $\beta + d\beta$ with the xz-plane. Further the components of velocity of molecule I will lie between u_1 and $u_1 + du_1$, v_1 and $v_1 + dv_1$ and w_1 and $w_1 + dw_1$, those of molecule II lying between u_2 and $u_2 + du_2$ etc. The chance that such components of velocity occur is represented for the two molecules respectively by

[1]) Counted in the direction of molecule I towards II.

$$\frac{1}{\pi\sqrt{\pi}}\, e^{-\frac{1}{\alpha^2}\{(u_1-az_1)^2+v_1^2+w_1^2\}}\, d\frac{u_1}{\alpha}\, d\frac{v_1}{\alpha}\, d\frac{w_1}{\alpha}$$

and

$$\frac{1}{\pi\sqrt{\pi}}\, e^{-\frac{1}{\alpha^2}\{(u_2-az_2)^2+v_2+w_2^2\}}\, d\frac{u_2}{\alpha}\, d\frac{v_2}{\alpha}\, d\frac{w_2}{\alpha}$$

in which $z_2 = z_1 + \sigma \cos\gamma$.

Hence the number of the collisions in question per second and per surface unity of the layer is:

$$\frac{n^2}{\pi^3}\, e^{-\frac{1}{\alpha^2}\{(u_1\ \ az_1)^2+v_1^2+w_1^2+(u_2-az_1-a\sigma\cos\gamma)^2+v_2^2+w_2^2\}}\, v_r \cos\mu\, \sigma^3 \sin\gamma\, d\gamma\, d\beta\, dz\ . \quad (3)$$

in which n denotes the number of molecules per cm³., v_r the relative velocity of molecule II with respect to molecule I, and μ the angle between the direction of v_r and the central line, so that:

$$v_r \cos\mu = (u_2-u_1)\sin\gamma\cos\beta + (v_2-v_1)\sin\gamma\sin\beta + (w_2-w_1)\cos\gamma$$

At each of these collisions the x-component of the quantity of motion, which is instantaneously transferred from above downwards is:

$$m\, v_r \cos\mu \sin\gamma\cos\beta\ .\ .\ .

Which integral is 1 for $-p < q < p$

and 0 for $q < -p$ and for $q > +p$.

Now if we put $p = s$ and $q = v_r \cos\mu + s$, and if we make s to increase indefinitely, the integral appears to become 1 for $v_r \cos\mu < 0$ and 0 for $v_r \cos\mu > 0$.

Thus we finally find for the force which the liquid above the plane $z = 0$ exerts per surface unity on that below it:

$$\eta a = -\frac{2n^2\sigma^3 m}{\pi^4}\int\{(u_2-u_1)\sin\gamma\cos\beta + (v_2-v_1)\sin\gamma\sin\beta + (w_2-w_1)\cos\gamma\}^3 \times$$
$$\times e^{-\frac{1}{\alpha^2}\{(u_1-az_1)^2+v_1^2+w_1^2+(u_2-az_1-a\sigma\cos\gamma)^2+v_2^2+w_2^2\}} \times$$
$$\times e^{+v\varphi\{(u_2-u_1)\sin\gamma\cos\beta+(v_2-v_1)\sin\gamma\sin\beta+(w_2-w_1)\cos\gamma+s\}} \times \frac{\sin s\varphi}{\varphi} \times \quad (5a)^1)$$
$$\times \sin^2\gamma\cos\beta\, d\gamma\, d\beta d\frac{u_1}{a}\ldots d\frac{w_2}{a}\, dz_1$$

As az_1 and az_2 will be in general very small compared with u_1 and u_2, we may write for the first exponential factor under the integral sign:

$$\left\{1 + \frac{1}{\alpha^2}2a\,(u_1z_1 + u_2z_1 + u_2\sigma\cos\gamma)\right\} e^{-\frac{1}{\alpha^2}(u_1^2+\ldots+w_2^2)}$$

When we substitute this in the integral, the term 1 between the accolades in (6) will furnish 0 after integration: it is the value of the force of friction for $a = 0$. The integrals with $\frac{2a}{\alpha^2}\, z_1\, u_1$ and $\frac{2a}{\alpha^2} z_1 u_2$ will become equal, but of opposite sign, so that they cancel each other, and the integral with $\frac{2a}{\alpha^2}\, u_2\, \sigma\cos\gamma$ only remains. When we now divide by a, and when we still put $\frac{z_1}{\sigma} = z'_1$ and $\frac{u_1}{a} = u'_1 \ldots\ldots \frac{w_2}{a} = w'_2$; $a\varphi = \varphi$ and $\frac{s}{a} = s'$, and when we then again omit the accents, we get:

[1]) The minus sign has been written for this, because $v_r \cos\mu$ is negative, while the number of collisions are naturally positive, and the sign of expression (3) should properly speaking be reversed.

$$\eta = \frac{-4n^2\sigma^4 m\alpha}{\pi^4}\int\{(u_2-u_1)\sin\gamma\cos\beta+(v_2-v_1)\sin\gamma\sin\beta+(w_2-w_1)\cos\gamma\}^2\times u_2\times$$
$$\times e^{-(u_1^2+\ldots+w_2^2)+i\varphi\{(u_2-u_1)\sin\gamma\cos\beta+(v_2-v_1)\sin\gamma\sin\beta+(w_2-w_1)\cos\gamma+s\}}\times \quad (5b)$$
$$\times \frac{\sin s\varphi}{\varphi}\sin^2\gamma\cos\gamma\cos\beta\, d\gamma\, d\beta\, d\varphi\, du_1\ldots dw_2\, dz_1$$

If we substitute in this

$$\left.\begin{aligned} u_1+\tfrac{1}{2}\iota\varphi\sin\gamma\cos\beta &= \xi_1 & u_2-\tfrac{1}{2}\iota\varphi\sin\gamma\cos\beta &= \xi_2\\ v_1+\tfrac{1}{2}\iota\varphi\sin\gamma\sin\beta &= \eta_1 & v_2-\tfrac{1}{2}\iota\varphi\sin\gamma\sin\beta &= \eta_2\\ w_1+\tfrac{1}{2}\iota\varphi\cos\gamma &= \zeta_1 & w_2-\tfrac{1}{2}\iota\varphi\cos\gamma &= \zeta_2 \end{aligned}\right\} \quad (6)$$

we get:

$$\eta = \frac{-4n^2\sigma^4 m\alpha}{\pi^4}\int\{(\xi_2-\xi_1)\sin\gamma\cos\beta+(\eta_2-\eta_1)\sin\gamma\sin\beta+(\zeta_2-\zeta_1)\cos\gamma+\iota\varphi\}^2\times$$
$$\times(\xi_2+\tfrac{1}{2}\iota\varphi\sin\gamma\cos\beta)\times e^{-(\xi_1^2\ldots\zeta_2^2)-\frac{1}{2}\varphi^2+\iota\varphi s}\times\frac{\sin s\varphi}{\varphi}\times \quad (5c)$$
$$\times\sin^2\gamma\cos\gamma\cos\beta\, d\varphi\, d\gamma\, d\beta\, d\xi_1\ldots d\zeta_2\, dz_1$$

On integration with respect to $\xi_1\ldots\zeta_2$ terms containing odd powers of $\xi_1\ldots\zeta_2$ vanish, so that the only terms left are those with $+\{[(\xi_2^2+\xi_1^2)\sin^2\gamma\cos^2\beta+(\eta_2^2+\eta_1^2)\sin^2\gamma\sin^2\beta+(\zeta_2^2+\zeta_1^2)\cos^2\gamma]\tfrac{1}{2}\iota\varphi\sin\gamma\cos\beta+$ $+2\xi_2^2\iota\varphi\sin\gamma\cos\beta-\tfrac{1}{2}\iota\varphi^3\sin\gamma\cos\beta\}$.

These terms do not change when ξ_2^2 is substituted for ξ_1^2, η_1^2, η_2^2, ζ_1^2, and ζ_2^2, so that $+(3\xi_2^2\iota\varphi-\tfrac{1}{2}\iota\varphi^3)\sin\gamma\cos\beta$ may be written for the sum of the remaining terms. After execution of the integrations we find:

$$\eta = -\frac{2}{\pi}n^2\sigma^4 m\alpha i\int(3-\varphi^2)\times e^{-\frac{1}{2}\varphi^2+\varphi s}\times\sin s\varphi\times$$
$$\times\sin^2\gamma\cos\gamma\cos^2\beta\, d\varphi\, d\gamma\, d\beta\, dz_1 \quad \ldots\ldots\ldots \quad (5d)$$

Let us now replace $e^{\iota\varphi s}$ by $\cos\varphi s+\iota\sin\varphi s$, and execute the integration with respect to φ, bearing in mind that we seek the value of the integral for $lim.\ s=\infty$. Then the term with $\sin s\varphi\cos s\varphi$ vanishes, and in the term with $\sin^2 s\varphi$ we may replace this expression by its mean value $\frac{1}{2}$. Thus we find:

$$-\iota\int_{-\infty}^{+\infty}e^{-\frac{1}{2}\varphi^2+\varphi s}\times\sin s\varphi\times(3-\varphi^2)\,d\varphi=\sqrt{2\pi}$$

$$\int_0^{2\pi}\cos^2\beta\, d\beta=\pi$$

$$\int_0^{Bg\cos(-z_1)} \sin^3\gamma\cos\gamma\,d\gamma = \tfrac{1}{4}\left[\sin^4\gamma\right]_0^{Bg\cos(-z_1)} = \tfrac{1}{4}(1-z_1^2)^2$$

$$\int_{-1}^{0} (1-z_1^2)^2\,dz_1 = \tfrac{8}{15}$$

$$\eta = \frac{4\sqrt{2\pi}}{15} n^2\sigma^4 m a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5e)$$

In the calculation of the number of collisions we have, however, up to now disregarded the influence of the mutual attraction of the molecules and of their dimension in the direction of the velocity. If for this we introduce the usual corrections, we find:

$$\eta = \frac{4\sqrt{2\pi}}{15} n^2\sigma^4 m a \frac{v}{v-b} e^{-\frac{\varepsilon}{RT}} \quad \ldots\ldots\ldots\ldots \quad (5)$$

in which ε represents the difference between the amount of potential energy that the molecules in the liquid possess on an average, and the amount which they possess at the moment of a collision.

G. JÄGER [1]) and M. BRILLOUIN [2]) had already derived expressions for the friction of liquids; JÄGER considers exclusively "friction in consequence of impact forces", whereas BRILLOUIN takes besides these also the friction by means of transport into consideration. The method of calculation differs somewhat from that followed above. The results at which they arrive, are in somewhat modified notation:

JÄGER $$\eta = \frac{\sigma\varrho s}{6\left(1-\sqrt[3]{\frac{b}{v}}\right)}$$

BRILLOUIN $$\eta = \tfrac{1}{3}\varrho s\left\{\alpha D + \beta\frac{\sigma}{2(D-\sigma)}\right\}$$

In this ϱ represents the density, s the mean velocity of the molecules, and D the mean distance of a pair of adjacent molecules. α and β are two unknown constants, which will not differ much from 1, and which have been introduced, because all kinds of approximations have been introduced into the calculation, which renders the numerical coefficients not entirely certain. The first term of BRILLOUIN's formula refers to transport, the second to impact forces. It seems to me that BRILLOUIN should also have corrected the first term for the "thickness" of the molecules. In his train of

[1]) G. JÄGER, Wiener Sitzungsber. CII, p. 253, Anno 1893.

[2]) M. BRILLOUIN, Leçons sur la Viscosité des Liquides et des Gaz. Paris. GAUTHIER-VILLARS 1907.

thought this might have been done by multiplication by a factor $\frac{D-\sigma}{D}$. That he failed to do so deprives his test of the experimental data of much of its value, in my opinion.

§ 3. *Test of the formulae for liquids not too near the critical point.* Let us call the "coefficient of friction by means of transport" η_1, that through forces of collision η_2, and that in consequence of formation of groups η_3. For liquids not too near the critical point we shall disregard η_3. We have further:

$$\frac{\eta_2}{\eta_1} = c\,\frac{b^3}{(v-b)^3}$$

in which c is a numerical coefficient of moderate value. We may no doubt consider this quantity as large compared with 1, so that we shall also neglect η_1.

When we do so we notice first of all that for constant volume η according to the formula must increase with T proportional to $\sqrt{T}$. There are only few substances for which the experimental data are available, required to verify whether the sign of $\frac{\partial \eta}{\partial T_v}$ is really positive. It is clear that always $\frac{\partial \eta}{\partial T_p}$ has been measured, and not $\frac{\partial \eta}{\partial T_v}$. Ether and Benzene are the only substances for which I have found records for $\frac{\partial \eta}{\partial p_T}$ [1]), so that $\frac{\partial \eta}{\partial T_v}$ can be found according to the formula

$$\frac{1}{\eta}\frac{\partial \eta}{\partial T_v} = \frac{1}{\eta}\frac{\partial \eta}{\partial T_p} + \frac{1}{\eta}\frac{\partial \eta}{\partial p_T}\frac{\partial p}{\partial T_v} = \frac{1}{\eta}\frac{\partial \eta}{\partial T_p} - \frac{1}{\eta}\frac{\partial \eta}{\partial p_T}\,\frac{\frac{1}{v}\frac{\partial v}{\partial T_p}}{\frac{1}{v}\frac{\partial v}{\partial p_T}}.$$

We find:

	for ether	for benzene
$\frac{1}{\eta_0}\frac{\partial \eta}{\partial T_p}$ =	$-0{,}01075$ [2])	$-0{,}01853$ [3])

[1]) Except for water, which will most likely also behave abnormally in this respect, and for CO_2 in the neighbourhood of the critical point, which observations will be discussed later on.

[2]) These values, like those of $\frac{1}{\eta_0}\frac{\partial \eta}{\partial T_p}$ for other substances given below have

$$\frac{1}{\eta_v}\frac{\partial \eta}{\partial p_T} = \quad 0{,}00073 \ ^{1}) \qquad 0{,}00093 \ ^{1})$$

$$\frac{1}{v}\frac{\partial v}{\partial T_p} = \quad 0{,}001585 \ ^{2}) \qquad 0{,}0011763 \ ^{3})$$

$$\frac{1}{v}\frac{\partial v}{\partial p_T} = -\,0{,}000139 \ ^{4}) \qquad -\,0{,}0000783 \ ^{5})$$

This yields for ether

$$\frac{1}{\eta_0}\frac{\partial \eta}{\partial T_v} = -\,0{,}01075 + 0{,}0088 = -\,0{,}00195$$

and for benzene

$$\frac{1}{\eta_0}\frac{\partial \eta}{\partial T_v} = -\,0{,}01853 + 0{,}01462 = -\,0{,}00391$$

Theoretically we should find according to equation (5):

$$\frac{1}{\eta_0}\frac{\partial \eta}{\partial T_v} = \frac{1}{2T} = +\,0{,}00183.$$

Accordingly there is not even agreement in the sign of $\frac{1}{\eta_0}\frac{\partial \eta}{\partial T_v}$. When we, however, take into consideration that the value found is the difference of two values which are each about five times the value of the amount sought, and that they are very inaccurately known, it is not excluded that $\frac{1}{\eta_0}\frac{\partial \eta}{\partial T_v}$ is in reality negative. Even in the value of $\frac{1}{v}\frac{\partial v}{\partial p_T}$ an error of 4 % is by no means excluded, and the error in the determination of $\frac{1}{\eta}\frac{\partial \eta}{\partial p_T}$ will without doubt be many times larger than that in $\frac{1}{v}\frac{\partial v}{\partial p_T}$. On the other hand it is of course

---

been found by differentiation of the empirical formulae of the form: $\eta = \frac{a}{(b+t)^c}$, in which THORPE and RODGER, Phil. Trans. Royal Soc. of London **185** p. 397, A. 1894 comprise their observations.

[1]) According to WARBURG and SACHS. Ann. d. Phys. u. Chem. **22** p. 521. A. 1884. The pressure is expressed in kg. per cm².

[2]) According to AMAGAT 1893, extrapolated for 0° and 1 atm. from the values given in the Recueil de constantes physiques.

[3]) According to KOPP. 1847. Borrowed from the "Recueil etc.".

[4]) According to SUCHODSKI 1910, extrapolated for 0° and 1 atm. from the values given in the "Recueil etc.".

[5]) According to RÖNTGEN 1891, extrapolated for 0° and 1 atm. from the values given in the "Recueil etc.".

also possible that even if formula (5) is valid, the value of $\frac{1}{\eta_0}\frac{\partial \eta}{\partial T_v}$ must be negative, in consequence of the factor $e^{-\frac{\varepsilon}{RT}}$, or because σ, hence also b depend on the temperature. The experimental data are not sufficiently accurate to decide this question.

More satisfactory results are furnished by another test, which can be applied on a more extensive scale. It consists in this that we compare the experimental values of $\frac{1}{\eta}\frac{\partial \eta}{\partial T_p}$ [1]) with the values following from equation (5).

For this purpose we write:

$$\frac{1}{\eta}\frac{\partial \eta}{\partial T_p} = \frac{1}{\eta}\left(\frac{\partial \eta}{\partial T_v} + \frac{\partial \eta}{dv_T}\frac{\partial v}{\partial T_p}\right)$$

and in this we put:

$$\frac{1}{\eta}\frac{\partial \eta}{\partial T_v} = \frac{1}{2T} \qquad \frac{1}{\eta}\frac{\partial \eta}{\partial v_T} = \frac{1}{v} - \frac{1}{v-b} = \frac{-b}{v(v-b)} = -\frac{b\left(p+\frac{a}{v^2}\right)}{v.RT.}$$

We shall neglect p by the side of $\frac{a}{v^2}$ and roughly assume $RT_k = \frac{8}{27}\frac{a}{b}$ and $p_k = \frac{1}{27}\frac{a}{b^2}$; we then get:

$$\frac{1}{\eta}\frac{\partial \eta}{\partial v_T} = -\frac{1}{RTv^3}\left(\frac{3}{8}\right)^3 \frac{R^3 T_k^3}{p^2 k}.$$

v is the volume per gram-molecule, hence $v = m\frac{1}{\varrho}$, so that we finally find to test:

$$\left(\frac{1}{\eta}\frac{\partial \eta}{\partial T_p}\right)_{exp} = \frac{1}{2T} - \frac{3^3 R^2 \varrho^2 T_k^3}{8^3 T\, m^2 p_k^2}\left(\frac{1}{v}\frac{\partial v}{\partial T_p}\right) \quad . \quad . \quad . \quad . \quad (7)$$

Borrowing the values of ϱ, p_k, T_k and $\frac{1}{v}\frac{\partial v}{\partial T_p}$ from the "Recueil etc.", we find: (See Table p. 754).

The agreement is on the whole as satisfactory as could be expected in view of the many approximations. Generally the experimental value is somewhat smaller than the theoretical one, for ether more than for other substances, benzene and orthoxylene deviating in the opposite sense. For acetic acid and for the alcohols the agreement is much less than for the normal substances.

---

[1]) See note 1 on pag. 751.

	$\left(\frac{1}{\eta}\frac{\partial\eta}{\partial T_p}\right)_{exp}$	$\left(\frac{1}{\eta}\frac{\partial\eta}{\partial T_p}\right)_{theor.}$	$\sigma \times 10^8$	$\sigma' \times 10^8$
Pentane	0.01019	0.01269	5.486	4.863
Isopentane	1081	1171	5.774	4.838
Hexane	1123	1354	6.716	5.166
Heptane	1214	1353	7.779	5.456
Octane	1394	1574	9.808	5.714
Chloroform	1149	1066	6.515	4.317
Ether	1075	1463	6.578	4.731
Benzolene	1853	1382	9.770	4.559
Toluolene	1462	1524	8.451	4.867
Orthoxylene	1700	1385	10.871	5.170
Metaxylene	1418	1478	7.973	5.223
Paraxylene [1])	1472	1414	6.716	5.187
Acetic acid [1])	1826	2607	4.713	4.382
Methyl alcohol	1634	1988	4.527	3.749
Ethyl „	2086	1250	10.273	4.046
Propyl „	2887	0970	36.103	4.421

So far we have tested the temperature-coefficients of η. We can also test equation (5) directly, namely by for instance calculating σ from it, and by comparing the values obtained thus with the values of σ calculated in another way. When we again omit the factor $e^{\frac{\varepsilon}{RT}}$ and when we put $\frac{2}{3}\pi\sigma^3 N = b = \frac{RT_k}{8p_k}$ (N = number of molecules per gram-molecule), substituting again $\frac{RT}{p+\frac{a}{v^2}}$ for $v-b$, and neglecting p by the side of $\frac{a}{v^2}$, we find:

[1]) For these substances the values for $\frac{1}{\eta_i}\frac{\partial\eta_i}{\partial T_p}$ also for 0° C. have been calculated from the empirical formula of RODGER and THORPE, though they are solid at this temperature.

$$\sigma = \frac{640}{27}\left(\frac{\pi T}{R^5}\right)^{1/2} \frac{m^{\frac{5}{2}} p_k^3}{\varrho^3 T_k^3}\,\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The values thus calculated for σ are recorded in the table on p. 754. With these the values calculated from the critical quantities:

$$\sigma' = \left(\frac{3RT_k}{16\pi N p_k}\right)^{1/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

have been compared. $N = 6{,}08 \times 10^{-23}$ (SOMMERFELD).

They are represented by σ' and recorded in the last column of the table. It appears that equation (8) gives values that are in perfect concordance with those of equation (9) as far as order of magnitude is concerned. It is noteworthy that the values for σ' differ little inter se, those for σ presenting much greater differences between each other. The alcohols show again great deviations.

Anatomy. — *"On two Nerves of Vertebrates agreeing in Structure with the Nerves of Invertebrates."* By Dr. A. B. Droogleever Fortuyn. (Communicated by Prof. J. Boeke).

(Communicated in the meeting of November 30, 1918).

As a well-known fact the olfactory cells in the mucous membrane of the nose of Vertebrates are "Sinnesnervenzellen" or "conducting sense-cells" as I propose to call them, unless they have received already another English name. (The word "sensory nerve-cell" may then be reserved to "sensibele Ganglienzelle"). They are sense-cells which are not surrounded by nerve-fibres, but whose cell-body directly passes into a process with all the characteristics of a nerve-fibre. These nerve-fibres, the fila olfactoria, constitute the nervus olfactorius. So the olfactory nerve deviates in its structure from all the other nerves of the vertebrated animals. The truth of this remark may already be deduced from the fact that in Vertebrates besides in the olfactory mucous membrane conductive sense-cells are only found in the retina (rod- and cone-cells) and perhaps in the pineal organ. So all the nerves of the Vertebrates with exception of the olfactory nerve are devoid of nervous processes of conducting sense-cells. On the contrary the majority of the nerves of invertebrated animals do contain processes of conductive sense-cells, which in these animals are always spread about the whole body in all kinds of sense-organs. Often in Invertebrates nerves are composed exclusively of fascicles of neurites of conducting sense-cells as is the case in the olfactory nerve of the Vertebrates. From this I conclude that we have to look upon the olfactory nerve of the Vertebrates as a nerve constructed in a way which is often met with in Invertebrates, and nowhere else in vertebrated animals.

Another striking difference between the nerves of vertebrated and invertebrated animals is this that in the Invertebrates ganglion-cells are generally dispersed along the whole course of the nerve either separately or in groups, ganglia. In the nerves of Vertebrates, however, they are totally wanting or they are accumulated in very few ganglia (spinal ganglia or those of the nerves of the brain). Now if one reads (e.g. in the review on this subject by Prof. van Wijhe in these Proceedings, Vol. XXVI, 1918) that also in the nervus

terminalis of the Vertebrates ganglion-cells are scattered along the whole course of the nerve, then one is compelled to grant to the terminal nerve the character of a nerve of an invertebrated animal. Nevertheless this type of nerve is obviously another than that of the olfactory nerve.

In connection with the preceding remarks I should like to point to the fact that Amphioxus, whose nervous system is generally compared with that of vertebrated animals, possesses in its sensory nerves the nerves of a true Invertebrate. All or most of them contain processes of conducting sense-cells as these cells are scattered about the whole body, and moreover most of the sensory nerves are accompanied by dispersed ganglion-cells. Therefore spinal ganglia are lacking.

Leiden, Anatomical Cabinet.

Mathematics. — "*Ueber die Teilkörper des Kreiskörpers* $k\left(e^{\frac{2\pi i}{l^h}}\right)$." (Zweiter Teil). Von Dr. N. G. W. H. BEEGER. (Communicated by Prof. W. KAPTEYN).

(Communicated in the meeting of October 26, 1918).

Wir beweisen zunächst einige Hülfssätze über die Function F und das Karaktersymbol.

Hülfssatz I.

$$F\left(e^{\frac{2k\pi i}{l^h}}\right) = (-1)^{bu} F\left(e^{-\frac{2k\pi i}{l^h}}\right).$$

Beweis:

$$F\left(e^{\frac{2k\pi i}{l^h}}\right) = \sum_{n=1}^{l^h-1}\left[\frac{n}{l^h}\right]^{bu} e^{\frac{2kn\pi i}{l^h}} =$$

$$= \sum_{n=1}^{l^h-1}\left[\frac{l^h-n}{l^h}\right]^{bu} e^{\frac{2k(l^h-n)i}{l^h}} = \sum_{n=1}^{l^h-1}\left[\frac{-n}{l^h}\right]^{bu} e^{-\frac{2kn\pi i}{l^h}}$$

$$= \left[\frac{-1}{l^h}\right]^{bu} \sum_{n=1}^{l^h-1}\left[\frac{n}{l^h}\right]^{bu} e^{-\frac{2kn\pi i}{l^h}}$$

$$= e^{\frac{2\pi l^{h-1}\left(\frac{l-1}{2}\right)bu\,i}{l^{h-1}(l-1)}} F\left(e^{-\frac{2k\pi i}{l^h}}\right)$$

$$= (-1)^{bu} F\left(e^{-\frac{2k\pi i}{l^h}}\right).$$

Hülfssatz II: Es sei $l^{h'}$ die höchste Potenz von l die in u aufgeht. Dann ist

$$\left[\frac{n+l^{h-h'}}{l^h}\right]^u = \left[\frac{n}{l^h}\right]^u.$$

Beweis: Es sei $n + l^{h-h'} \equiv r^{n'} \pmod{l^h}$
dann ist

$$n \equiv r^{n'} \pmod{l^{h-h'}} \text{ wegen } h' < h$$

Es sei nun

$$n \equiv r^{m'} \pmod{l^h} \text{ also auch } \pmod{l^{h-h'}}$$

dann ist

$$r^{m'} \equiv r^{n'} \pmod{l^{h-h'}}$$

also

$$m' \equiv n' \,(mod\, l^{h-h'-1}(l-1))$$

oder

$$n' = m' + l^{h-h'-1}(l-1)v.$$

Hieraus ergibt sich:

$$\left[\frac{n+l^{h-h'}}{l^h}\right]^u = e^{\frac{2\pi n'ui}{l^{h-1}(l-1)}} = e^{\frac{2\pi m'ui}{l^{h-1}(l-1)} + \frac{2\pi l^{h-h'-1}(l-1)ui}{l^{h-1}(l-1)}}$$

$$= \left[\frac{n}{l^h}\right]^u \cdot e^{\frac{2\pi u l^{h-h'-1}(l-1)i}{l^{h-1}(l-1)}}.$$

Der letzte Factor ist $= 1$ da u teilbar ist durch $l^{h'}$.

Hülfssatz III. Es sei $l^{h'}$ die höchste Potenz von l die auf u teilbar ist, so ist:

$F\left(e^{\frac{2k\pi i}{l^h}}\right) = 0$ wenn die höchste Potenz von l, die auf k teilbar ist, $\neq l^{h'}$ ist

$$= \left[\frac{k/l^{h'}}{l^h}\right]^{-bu} F\left(e^{\frac{2\pi i}{l^{h-h'}}}\right)$$ wenn die letzt genannte höchste Potenz $= l^{h'}$ ist.

Beweis: Es folgt aus Hülfssatz II:

$$F\left(e^{\frac{2k\pi i}{l^h}}\right) = \sum_{n=1}^{l^h-1}\left[\frac{n}{l^h}\right]^{bu} e^{\frac{2kn\pi i}{l^h}} = \sum_{n=1}^{l^h-1}\left[\frac{n+l^{h-h'}}{l^h}\right]^{bu} e^{\frac{2kn\pi i}{l^h}}$$

$$= e^{\frac{-2k\pi i}{l^{h'}}} \sum_{n=1}^{l^h-1}\left[\frac{n+l^{h-h'}}{l^h}\right]^{bu} e^{\frac{2k(n+l^{h-h'})\pi i}{l^h}}$$

$$= e^{\frac{-2k\pi i}{l^{h'}}} F\left(e^{\frac{2k\pi i}{l^h}}\right)$$

also ist

$$\left(1 - e^{\frac{2k\pi i}{l^{h'}}}\right) F\left(e^{\frac{2k\pi i}{l^h}}\right) = 0$$

Wenn nun die höchste Potenz von l, welche in k aufgeht, kleiner ist als $l^{h'}$, so ergibt sich hieraus $F = 0$. Nehmen wir nun weiter an dasz k teilbar ist durch eine höhere Potenz von l, $l^{h'+c}$, $c > o$, dann ist:

$$F\left(e^{\frac{2k\pi i}{l^h}}\right) = \sum_{n=1}^{l^h}\left[\frac{n}{l^h}\right]^{bu} e^{\frac{2kn\pi i}{l^h}} = \sum_{n=1}^{l^{h-h'-c}} + \sum_{n=l^{h-h'-c}+1}^{2l^{h-h'-c}} + \ldots + \sum_{n=(l^{h'+c}-1)\,l^{h-h'-c}+1}^{l^h}$$

Nun ist

$$e^{\frac{2kn\pi i}{l^h}} = e^{\frac{2k'n\pi i}{l^{h-h'-c}}} \quad \text{wegen} \quad k = l^{h'+c}\,k'$$

also:

$$F\left(e^{\frac{2k\pi i}{l^h}}\right) = \sum_{n=1}^{l^{h-h'-c}} e^{\frac{2k\pi ni}{l^h}} \sum_{s=0}^{l^{h'+c}-1} \left[\frac{n + sl^{h-h'-c}}{l^h}\right]^{bu}$$

Es sei

$$n + sl^{h-h'-c} \equiv r^{s'} \pmod{l^h} \quad \text{und} \quad n \equiv r^{n'} \pmod{l^h}$$

dann ist

$$n \equiv r^{s'} \pmod{l^{h-h'-c}} \qquad n \equiv r^{n'} \pmod{l^{h-h'-c}}$$

also

$$r^{s'} \equiv r^{n'} \pmod{l^{h-h'-c}}$$
$$s' \equiv n' \bmod \varphi(l^{h-h'-c})$$

und

$$s' = n' + v_s \varphi(l^{h-h'-c}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hieraus folgt:

$$\left[\frac{n + sl^{h-h'-c}}{l^h}\right]^{bu} = e^{2\pi i \frac{n' + v_s \varphi(l^{h-h'-c})}{l^{h-1}(l-1)} bl^{h'}u}$$

(wo $u = l^{h'}u'$)

$$= e^{\frac{2\pi i n' bu}{l^{h-1}(l-1)}} \cdot e^{2\pi i \frac{v_s bu'}{l^c}}$$

$$= \left[\frac{n}{l^h}\right]^{bu} e^{2\pi i \frac{v_s bu'}{l^c}}$$

und schliesslich

$$F\left(e^{\frac{2k\pi i}{l^h}}\right) = \sum_{n=1}^{l^{h-h'-c}} \left[\frac{n}{l^h}\right]^{bu} e^{\frac{2k\pi ni}{l^h}} \sum_{s=0}^{l^{h'+c}-1} e^{2\pi i \frac{v_s bu'}{l^c}} \quad . \quad . \quad (6)$$

Wenn nun in der letzten Summe zwei Werte von $v_s \pmod{l^c}$ mit einander congruent sind, so ergeben diese Werte zwei gleiche Glieder dieser Summe. Nehmen wir an dasz:

$$v_{s_1} \equiv v_{s_2} \pmod{l^c}$$
$$v_{s_1} = v_{s_2} + wl^c.$$

und dasz weiter

$$s'_1 = n' + v_{s_1}\varphi(l^{h-h'-c}) \quad \text{und} \quad s'_2 = n' + v_{s_2}\varphi(l^{h-h'-c})$$

dann ist auch

$$s_1' = n' + (v_{s_2} + wl^c)\,\varphi(l^{h-h'-c})$$
$$s'_1 = n' + v_{s_2}\varphi(l^{h-h'-c}) + wl^c\varphi(l^{h-h'-c})$$
$$s'_1 = s'_2 + wl^c\varphi(l^{h-h'-c}).$$

Weiter ist:

$$n + s_1 l^{h-h'-c} \equiv r^{s_1'} \quad \text{und} \quad n + s_2 l^{h-h'-c} \equiv r^{s_2'} \pmod{l^h}$$

und auch ist also:

$$n + s_1 l^{h-h'-c} \equiv r^{s_2'} + w l^c \varphi(l^{h-h'-c})$$

$$\equiv r^{s_2'} \cdot r^{w l^c \varphi(l^{h-h'-c})}$$

$$\equiv (n + s_2 l^{h-h'-c}) \cdot r^{w l^c \varphi(l^{h-h'-c})}$$

und wegen

$$r^{l^c \varphi(l^{h-h'-c})} = r^{l^{h-h'-1}(l-1)} \equiv 1 \pmod{l^{h-h'}},$$

folgt hieraus

$$n + s_1 l^{h-h'-c} \equiv n + s_2 l^{h-h'-c} \pmod{l^{h-h'}}$$

also

$$s_1 \equiv s_2 \pmod{l^c}$$

Schliesslich bemerken wir noch dasz die Werte von v_s, die zu zwei verschiedenen Gliedern der Summe $\sum_s$ aus (6) gehören, nicht gleich sind; denn, nehmen wir an dasz die zwei Werte von v, die zu s_1 und s_2 gehören, gleich sind, so folgt aus (5) $s_1' = s_2'$, und also würde

$$n + s_1 l^{h-h'-c} \equiv r^{s'} \text{ en } n + s_2 l^{h-h'-c} \equiv r^{s'}$$

also

$$s_1 l^{h-h'-c} \equiv s_2 l^{h-h'-c} \pmod{l^h}$$

$$s_1 \equiv s_2 \pmod{l^{h'+c}}$$

Und dies ist nicht möglich da s in der Summe $\sum$ nur läuft bis $s = l^{h'+c} - 1$.

Hieraus folgt dasz $\sum_s = \frac{l^{h'+c}}{l^c} \times$ die Summe aller l^c-ten Einheitswurzeln.

Diese letzte Summe ist aber, wie bekannt, gleich Null, so dasz sich ergibt $\sum_s = 0$ und $F = 0$.

Der Beweis des zweiten Teiles gestaltet sich wie folgt:

$$F\left(e^{\frac{2k\pi i}{l^h}}\right) = \sum_n \left[\frac{n}{l^h}\right]^{bu} e^{\frac{2kn\pi i}{l^h}} = \left[\frac{k/l^{h'}}{l^h}\right]^{-bu} \sum_n \left[\frac{nk/l^{h'}}{l^h}\right]^{bu} e^{\frac{2kni}{l^h}}$$

da $k/l^{h'}$ nicht teilbar ist durch l. Hieraus ergibt sich weiter

$$F\left(e^{\frac{2k\pi i}{l^h}}\right) = \left[\frac{k/l^{h'}}{l^h}\right]^{-bu} \sum_n \left[\frac{nk/l^{h'}}{l^h}\right]^{bu} e^{\frac{2nk/l^{h'}\pi i}{l^{h'-h'}}}.$$

Nun durchläuft $nk/l^{h'}$ zugleich mit n ein reducirtes Restsystem $(mod\, m)$, folglich ist:

$$F\left(e^{\frac{2k\pi i}{l^h}}\right) = \left[\frac{k/l^{h'}}{l^h}\right]^{-bu} \sum_n \left[\frac{n}{l^h}\right]^{bu} e^{\frac{2n\pi i}{l^{h-h'}}}$$

Hülfssatz IV.

$$F'\left(e^{\frac{2\pi i}{l^{h-h'}}}\right) F'\left(e^{\frac{2\pi i}{l^{h-h'}}}\right) = l^{h'+h}(-1), \quad b\frac{u}{l^{h'}}\, u \equiv \tfrac{1}{2}\, a l^{h-1}$$

wo $l^{h'}$ die höchste Potenz von l ist, die in u aufgeht, und F' aus der Function F erhalten wird, wenn in letzterer u durch $-u$ ersetzt wird.

Ist $u = \frac{1}{2}\, a l^{h-1}$, a gerade, so ist $F(e^{\frac{2\pi i}{l}}) = l^{h-1/2} i^{1/4(l-1)^2}$

Beweis: Es ist

$$F\left(e^{\frac{2\pi i}{l^{h-h'}}}\right) = \sum_{n=1}^{l^h-1} \left[\frac{n}{l^h}\right]^{bu} e^{\frac{2\pi ni}{l^{h-h'}}} = \sum_{n=1}^{l^h-1} \left[\frac{n^k}{l^h}\right]^{bu} e^{\frac{2\pi nki}{l^{h-h'}}}$$

wenn k nicht teilbar ist durch l. Hieraus geht hervor:

$$\left[\frac{k}{l^h}\right]^{-bu} F\left(e^{\frac{2\pi i}{l^{h-h'}}}\right) = \sum_{n=1}^{l^h} \left[\frac{n}{l^h}\right]^{bu} e^{\frac{2\pi kni}{l^{h-h'}}}$$

Wir multipliciren mit $e^{\frac{2\pi ki}{l^{h-h'}}}$ und nehmen die Summe über alle Werte von k, die nicht durch l teilbar sind. Es ergibt sich

$$F'\left(e^{\frac{2\pi i}{l^{h-h'}}}\right) F\left(e^{\frac{2\pi i}{l^{h-h'}}}\right) = \sum_{n=1}^{l^h} \left[\frac{n}{l^h}\right]^{bu} \sum_k e^{\frac{2\pi k(n+1)i}{l^{h-h'}}}.$$

Wir teilen die letzte Summe in drei Partialsummen:

1°. Die Summe über alle Werte von n für welche $n+1$ teilbar ist durch $l^{h-h'}$.

2°. Die Summe über alle Werte von n für welche $n+1$ teilbar ist durch $l^{h-h'-1}$ und nicht durch eine höhere Potenz von l.

3°. Die Summe über alle Werte von n für welche $n+1$ durch eine niedrigere Potenz von l teilbar ist, inclusiv die nullte-Potenz.

Bei dieser Einteilung haben wir keine Zahl n vergessen. Wir betrachten nun jede dieser drei Summen.

1°. In diesem Falle ist jedes Glied von $\sum_k$ gleich Eins, also $\sum_k = \varphi(l^h)$. Die hierher gehörigen Zahlen $n+1$ sind:

$$l^{h-h'},\quad 2l^{h-h'}, \ldots\ldots l^{h'}.l^{h-h'}$$

Für die Zahlen n ergibt sich hieraus

$$n = -1 + l^{h-h'} v_n \qquad v_n = 1, 2, \ldots\ldots l^{h'}$$

also:

$$\left[\frac{n}{l^h}\right]^{bu} = e^{\frac{2\pi bun'i}{\varphi(l^h)}} \qquad n \equiv r^{n'} \pmod{l^h}.$$ Dann ist aber

$$-1 \equiv r^{n'} \pmod{l^{h-h'}}$$

und also:

$$n' \equiv 0 \pmod{\tfrac{1}{2}\varphi(l^{h-h'})}$$

Folglich:

$$\left[\frac{n}{l^h}\right]^{bu} = e^{2\pi i \frac{bl^{h'}u' \cdot \frac{1}{2}\varphi(l^{h-h'})w_n}{\varphi(l^h)}} = (-1)^{bu'}$$

In der Summe 1°. finden sich $l^{h'}$ Glieder vor.
Infolge dessen ist der Wert dieser Summe:

$$(-1)^{bu'} l^{h'} \varphi(l^h)$$

2°. In diesem Falle ist:

$$e^{\frac{2\pi k(n+1)i}{l^{h-h'}}} = \text{einer } l\text{-ten Einheitswurzel.}$$

Die hierher gehörige Summe $\sum_k$ ist also gleich $\frac{\varphi(l^h)}{\varphi(l)} \times$ die Summe der primitiven l-ten Einheitswurzeln, d.h. gleich $-\frac{\varphi(l^h)}{\varphi(l)}$.

Weiter ist $\left[\frac{n}{l^h}\right]^{bu} =$ einer l-ten Einheitswurzel von ± 1 wegen

$$n = -1 + l^{h-h'-1} v_n$$

wo v_n alle Werte annimmt welche $< l^{h'+1}$ sind und nicht teilbar durch l. Das sind $\varphi(l^{h'+1})$ Werte.

Auf gleiche Weise wie unter 1°. zeigt man dasz $\left[\frac{n}{l^h}\right]^{bu}$ eine l-te Einheitswurzel ist aus 1 oder -1 je nachdem bu' gerade oder ungerade ist, wo $u = l^{h'}u'$. Jede primitive l-te Einheitswurzel tritt in der Summe $\sum_n \frac{\varphi(l^h)}{\varphi(l)} \times$ auf.

Gemäsz diesen Bemerkungen findet man leicht dasz der Wert der hierher gehörigen Partialsumme $\sum_n$ ist:

$$-\frac{\varphi(l^h)}{\varphi(l)} \times \mp \frac{\varphi(l^{h'+1})}{\varphi(l)} = l^{h'} \cdot l^{h-1} (-1)^{bu'}.$$

3°. Die Summe $\sum_k$ kan man verteilen in einige Male die Summe

der primitiven $l^{h-h'-c}$-te Einheitswurzeln, wo $c > 1$. Daher ist auch $\Sigma = 0$.
n

Nach einer kleinen Rechnung findet man nun den ersten Teil des Hülfssatzes.

Beweis des zweiten Teils:

$$F\left(e^{\frac{2\pi i}{l}}\right) = \sum_n \left[\frac{n}{l^h}\right]^{1/2 a b l^{h-1}} e^{\frac{2\pi ni}{l}} =$$

$$= \sum_n (-1)^{indn} e^{\frac{2\pi ni}{l}} = \sum_{t=1}^{\varphi(l^h)} (-1)^t e^{\frac{2\pi r_t i}{l}}$$

wo r_t den kleinsten positiven Rest bezeichnet von r^t (mod l)

$$F\left(e^{\frac{2\pi i}{l}}\right) = \sum_{t=1}^{l-1} + \sum_{t=l}^{2(l-1)} + \ldots.$$ Nun ist $r_{t+(l-1)} = r_t$ also

$$= l^{h-1} \cdot \sum_{t=1}^{l-1} (-1)^t e^{\frac{2\pi r_t i}{l}}$$

Die letzte Summe verteilt man in zwei Teile, n.l. für t gerade und t ungerade. Nun ist r_{2t} immer quadratischer Rest, und r_{2t+1} quadratischer Nichtrest. Man findet leicht dasz die Summe eine sogenannte Gaus'sche ist, deren Wert

$$l^{1/2}\, i^{1/4(l-1)^2}$$

Es ergibt sich nun aus (3) dasz die Klassenanzahl H (welche im Ersten Teil dieses Aufsatzes durch h angedeutet ist) $\frac{1}{x}$ Mal dem Producte über $u = 1, 2, \ldots . al^{h-1}-1$ von (4) gleich ist. Die weitere Herleitung der Formel für die Klassenanzahl, fällt nun sehr verschieden aus, je nachdem der Relativgrad b gerade oder ungerade ist. Es ist ersichtlich dasz im ersteren Falle der Teilkörper reell ist, und im lezteren imaginär.

1°. *Die Klassenanzahl für reelle Teilkörper (b gerade).*

Aus Hülfssatz I ergibt sich

$$F\left(e^{\frac{2k\pi i}{l^h}}\right) = F\left(e^{\frac{-2k\pi i}{l^h}}\right)$$

Weiter ist:

$$\sum_{k=1}^{l^h-1} kF\left(e^{\frac{2\pi ki}{l^h}}\right) = \sum_{k=1}^{l^h-1} (l^h-k) F\left(e^{\frac{2\pi(l^h-k)i}{l^h}}\right)$$

$$= \sum_{k=1}^{l^h-1} l^h F\left(e^{\frac{-2\pi ki}{l^h}}\right) - \sum_{k=1}^{l^h-1} kF\left(e^{\frac{-2\pi ki}{l^h}}\right)$$

$$= lh \sum_{k=1}^{l^h-1} F\left(e^{\frac{2\pi ki}{l^h}}\right) - \sum_{k=1}^{l^h-1} kF\left(e^{\frac{2\pi ki}{l^h}}\right)$$

Man findet daher:

$$2 \sum_{k=1}^{l^h-1} kF\left(e^{\frac{2\pi ki}{l^h}}\right) = lh \sum_{k=1}^{l^h-1} F\left(e^{\frac{2\pi ki}{l^h}}\right)$$

Es ist aber:

$$\sum_{k=1}^{l^h-1} F\left(e^{\frac{2\pi ki}{l^h}}\right) = \sum_{k=1}^{l^h-1} \sum_{n=1}^{l^h-1} \left[\frac{n}{l^h}\right] bu \left(e^{\frac{2\pi kni}{l^h}}\right)$$

$$= \sum_{n=1}^{l^h-1} \left[\frac{n}{l^h}\right] bu \sum_{k=1}^{l^h-1} e^{\frac{2\pi kni}{l^h}}$$

$$= \sum_{n=1}^{l^h-1} \left[\frac{n}{l^h}\right] bu \, . \, -1 = 0$$

Also ist:

$$\sum_{k} k F\left(e^{\frac{2\pi ki}{l^h}}\right) = 0.$$

und:

$$H = \frac{1}{\varkappa} \prod_{u}' - \frac{1}{l^h} \left[\sum_{k=1}^{l^h-1} F\left(e^{\frac{2\pi ki}{l^h}}\right) log \frac{e^{\frac{k\pi i}{l^h}} - e^{-\frac{k\pi i}{l^h}}}{i} \right]$$

oder

$$H = \frac{(-1)^{c-1}}{\varkappa l^h (a l^{h-1} - 1) \cdot} \prod_{u}' \sum_{k=1}^{l^h-1} F\left(e^{\frac{2\pi ki}{l^h}}\right) log A_k$$

wo:

$$A_k = \frac{e^{\frac{k\pi i}{l^h}} - e^{-\frac{k\pi i}{l^h}}}{i} = i \, e^{-\frac{k\pi i}{l^h}} \left(1 - e^{\frac{2k\pi i}{l^h}}\right)$$

Das Product zerlegen wir in verschiedene andere Producte:

1. Ueber alle Werte von u die nicht durch l teilbar sind,
2. Ueber alle Werte von u die durch l teilbar sind aber nicht durch l^2

. . . . . .

schliesslich über alle Werte von u die durch l^{h-1} teilbar sind.

Fassen wir nun ins Auge die Werte von u welche teilbar sind durch $l^{h'}$ und nicht durch $l^{h'+1}$.

Wegen Hülfssatz III finden wir

$$\sum_{k=1}^{l^h-1} F\left(e^{\frac{2\pi k i}{l^h}}\right) \log A_k = F\left(e^{\frac{2\pi i}{l^{h-h'}}}\right) \sum_k \left[\frac{k/l^{h'}}{l^h}\right]^{-ub} \log A_k$$

wo, in der Summe rechter Hand, k alle Zahlen durchläuft $< l^h$, welche teilbar sind durch $l^{h'}$ aber nicht durch $l^{h'+1}$. Diese Summe veränderen wir in eine Summe welche sich über alle Werte von k erstreckt, die nicht durch l teilbar sind. Es sei $k = l^{h'} . k'$, so ist

$$A_k = i e^{\frac{k\pi i}{l^h}\left(1 - e^{\frac{2k\pi i}{l^h}}\right)} = i e^{\frac{k\pi i}{l^h}\left(1 - e^{\frac{2\pi k' i}{l^h}}\right)\left(1 - e^{\frac{2\pi(k' + l^{h-h'})}{l^h}}\right) \cdots}$$

$$\cdots \left(1 - e^{\frac{2\pi(k' + (l^{h'}-1) l^{h-h'}) i}{l^h}}\right)$$

$$i e^{-\frac{k\pi i}{l^h}} = i l^{h'} e^{-\frac{\pi k' i}{l^h}} . e^{-\frac{\pi(k'+l^{h-h'})i}{l^h}} \cdots e^{-\frac{\pi(k'+(l^{k'}-1) l^{h-h'})i}{l^h}}$$

Hieraus folgt:

$$A_k = A_{k'} . A_{k'+l^{h-h'}} . A_{k'+2l^{h-h'}} \cdots A_{k'+(l^{h'}-1)l^{h-h'}}$$

Es ist also:

$$\sum_k \left[\frac{k/l^{h'}}{l^h}\right]^{-bu} \log A_k = \sum_k \left[\frac{k'}{l^h}\right]^{-bu} (\log A_{k'} + \log A_{k'+l^{h-h'}} + A_{k+(l^{h'}-1)l^{h-h'}})$$

und wegen Hülfssatz II:

$$= \sum_{k'} \left[\frac{k'}{l^h}\right]^{-bu} \log A_{k'} + \sum_{k'} \left[\frac{k'+l^{h-h'}}{l^h}\right]^{-bu} \log A_{k'+l^{h-h'}} + \cdots$$

$$\cdots + \sum_{k'} \left[\frac{k'+(l^{h'}-1)l^{h-h'}}{l^h}\right]^{-bu} \log A_{k'+(l^{h'}-1)l^{h-h'}}$$

Die ursprüngliche Summe dehnt sich aus über $(l-1)l^{h-h'-1}$ Werte wie man leicht findet. Diese Summe ist nun ersetzt durch $l^{h'}$ andere und kann daher aufgefasst werden als eine Summe über

$$l^{h'} . (l-1) l^{h-h'-1} = (l-1) l^{h-1}$$

Zahlen. Und diese Anzahl ist gleich der Anzahl der Zahlen welche $< l^h$ sind und nicht teilbar durch l. Es sind weiter nicht zwei dieser Zahlen $(mod\ l^h)$ mit einander congruent; denn wäre z. b.

$$k' + il^{h-h'} \equiv k' + jl^{h-h'} \ (mod\ l^h)$$

so würde sich ergeben:

$$i \equiv j \ (mod\ l^{h'})$$

und dies ist unmöglich da i und j beide $< l^{h'}$ sind.

Schliesslich ist also:

$$\sum_k \left[\frac{k/l^{h'}}{l^h}\right]^{-bu} log\, A_k = \sum_k \left[\frac{k}{l^h}\right]^{-bu} log\, A_k$$

wo, in der Summe rechter Hand, die Zahl k alle Werte durchläuft $< l^h$ welche nicht durch l teilbar sind.

Wir haben nun gefunden:

$$H = \frac{(-1)^{c-1}}{\varkappa\, l^{h(al^{h-1}-1)}} \prod_{u_1} F\left(e^{\frac{2\pi i}{l^h}}\right) . \prod_{u_2} F\left(e^{\frac{2\pi i}{l^{h-1}}}\right) \ldots . \prod_{u_{lh}} F\left(e^{\frac{2\pi i}{l}}\right) \times$$

$$\times \prod_u{}' \sum_k \left[\frac{k}{l^h}\right]^{-bu} log\, A_k.$$

Das erste Product erstreckt sich über alle Werte von u die auf der vorletzten Seite unter 1° genannt sind u. s. w. Fassen wir nun insbesondere das Product ins Auge, welches läuft über alle Werte von u welche teilbar sind durch $l^{h'}$ und nicht durch $l^{h'+1}$. Wir nehmen die Factoren zu zweien und gebrauchen Hülfssatz IV. Dies ist möglich wegen

$$\left[\frac{n}{l^h}\right]^{-bu} = \left[\frac{n}{l^h}\right]^{b(al^{h-1}-u)}$$

und da u und $al^{h-1}-u$ beide zugleich durch $l^{h'}$ teilbar sind. Man findet dann für jedes Factorenpaar: $l^{h+h'}$. Die Factorenzahl wird gleich der halben Anzahl der Zahlen welche $< al^{h-1}$ sind, und die durch $l^{h'}$ teilbar sind. Also

$$\tfrac{1}{2}(l-1)\, al^{h-h'-2}$$

Das Product, welches wir ins Auge gefasst haben, hat daher den Wert

$$l^{\frac{1}{2}(h+h')(l-1)al^{h-h'-2}}$$

Für $h' = 0, 1, \ldots . h-2$.

Im Falle $h' = h-1$ gibt es $a-1$ Factoren, denn es gibt $a-1$ Zahlen $< al^{h-1}-1$ welche durch l^{h-1} teilbar sind. Der Wert dieses Productes ist also

$$l^{(2h-1)\frac{a-2}{2}}$$

Der Endwert aller Producte, wird nun

$$l^{\frac{1}{2}hl-1)al^{h-2}+\frac{1}{2}(h+1)(l-1)al^{h-3}+\ldots+\frac{1}{2}(2h-2)(l-1)a+(2h-1)\frac{a-1}{2}}$$

$$= l^{\frac{1}{2}hal^{h-1}+\frac{1}{2}a\frac{l^{h-1}-1}{l-1}-h+\frac{1}{2}}$$

und für die Klassenanzahl findet man

$$H=\frac{(-1)^{al^{h-1}-1}}{\frac{1}{2}h\,al^{h-1}-\frac{1}{2}a\frac{l^{h-1}-1}{l-1}-\frac{1}{2}}\cdot\frac{1}{\varkappa l}-\prod_{u}{}'\sum_{k=1}^{l^h}\left[\frac{k}{l^h}\right]^{bu}\log A_k$$

Für jeden Körper gilt[1]:

$$\frac{1}{\varkappa}=\frac{w|\sqrt{D}|}{2^{r_1+r_2}\pi^{r_2}R}$$

In unserem Falle ist $w=2$, denn der Körper ist reell und es sind daher ± 1 die einzigen Einheitswurzeln welche sich im Körper vorfinden. $r_2=0$, $r_1=al^{h-1}$. Die Discriminante D ist bestimmt in Theorem 5. R ist der Regulator. Man findet, indem man dieses alles substituirt, dasz die Potenz von l verschwindet und

$$H=\frac{(-1)^{al^{h-1}-1}}{2^{al^{h-1}-1}}\cdot\frac{\prod_{u=1}^{al^{h-1}-1}\sum_{k=1}^{l^h}\left[\frac{k}{l^h}\right]^{bu}\log A_k}{R}\quad\ldots\quad(7)$$

Das Product is gleich einer Determinante, welche man findet durch ein bekanntes Theorem der Norm einer Algebraïschen Zahl[2]:

$$\sum_{k}\left[\frac{k}{l^h}\right]^{bu}\log A_k=\sum e^{\frac{2\pi bu\,ind\,k}{l^{h-1}(l-1)}}\log A_k$$

wo k alle Zahlen durchläuft $<l^h$ und die nicht durch l teilbar sind. Es nimmt also $ind\,k$ alle Werte an von 1 bis $\varphi(l^h)$. Setzen wir also $ind\,k=t$, so ist $r^t\equiv k\,(mod\,l^h)$.

Die letzte Summe nimmt daher folgende Form an:

$$\sum_{t=1}^{\varphi(l^h)}e^{\frac{2\pi bu\,ti}{l^{h-1}(l-1)}}\log A_{r^t}=\sum_{t=1}^{\varphi(l^h)}e^{\frac{2\pi uti}{al^{h-1}}}\log A_{r^t}$$

$$=\sum_{t=1}^{al^{h-1}}+\sum_{t=al^{h-1}+1}^{2al^{h-1}}+\ldots\ldots+\sum_{t=(b-1)al^{h-1}+1}^{bal^{b-1}}$$

$$=\sum_{t=1}^{al^{h-1}}+\sum_{t=1}^{al^{h-1}}e^{\frac{2\pi uti}{al^{h-1}}}\log A_{r^t+al^{h-1}}+\ldots\ldots+\sum^{al^{h-1}}e^{\frac{2\pi uti}{al^{h-1}}}\log A_{r^t+(b-1)al^{h-1}}$$

$$=\sum_{t=1}^{al^{h-1}}e^{\frac{2\pi uti}{al^{h-1}}}\log A_{r^t}\cdot A_{r^t+al^{h-1}}\ldots\ldots A_{r^t+(b-1)al^{h-1}}$$

---

[1]) H. Seite 229.

[2]) Baltzer. Th. u. Anw. der Determinanten. S. 98.

Es sei jetzt:

$$A_t = e^{\frac{\pi i}{l^h}(r^t + \ldots + r^{t+(b-1)al^{h-1}})} i^b (1-Z^{r^t})(1-Z^{r^{t+al^{h-1}}}) \ldots (1-Z^{r^{t+(b-1(al^{h-1})}})$$

$$= i^b \cdot e^{\pi i \cdot r^t \cdot g} \cdot (1-Z^{r^t})(1-Z^{r^{t+al^{h-1}}}) \ldots (1-Z^{r^{t+(b-1)al^{h-1}}}).$$

So ist: (wo g eine ganze Zahl bedeutet)

$$\prod_{u=1}^{al^{h-1}-1} \sum_{t=1}^{al^{h-1}} e^{\frac{2\pi uti}{al^{h-1}}} \log A_t = \prod_{u=1}^{al^{h-1}-1} e^{\frac{2\pi ui}{al^{h-1}}} \sum_{t=1}^{al^{h-1}} e^{\frac{2\pi u(t-1)i}{al^{h-1}}} \log A_t =$$

$$= \frac{e^{\frac{2\pi i}{al^{h-1}}(1+2+\ldots+al^{h-1}-1)}}{\log A_1 + \ldots + \log A_{al^{h-1}}} \begin{vmatrix} \log A_1 & \log A_2 \ldots \log A_{al^{h-1}} \\ \log A_{al^{h-1}} & \log A_1 \ldots \log A_{al^{h-1}-1} \\ \ldots & \ldots \\ \log A_2 & \log A_3 \ldots \log A_1 \end{vmatrix}$$

Man kan diese Determinante auf folgende Weise umformen. Von den Elementen einer jeden Colonne werden die entsprechenden Elemente der vorangehenden Colonne subtrahirt, wobei man mit der vorletzten anfängt, und schiesslich von den Elementen der ersten Colonne die entsprechenden Elemente der letzten Colonne subtrahirt. Jetzt addirt man die entsprechenden Elemente einer jeden Zeile zu denen der letzten. Alle Elemente dieser letzten Zeile verschwinden dann, ausgenommen das letzte, welches folgenden Wert annimmt:

$$\log A_1 + \log A_2 + \ldots + \log A_{al^{h-1}}.$$

Dies ist auch der Factor der sich im Nenner des Bruches vor der Determinante findet. Er fällt also hinweg. Die Elemente der Determinante haben nun folgende Form erhalten:

$$\log \frac{A_t}{A_{t-1}} = \log e^{\frac{\pi i}{l^h}(r^t + r^{t+c} + \ldots + r^{t+(b-1)c}) - (r^{t-1} + \ldots)} \times$$

$$\times \frac{(1-Z^{r^t})(1-Z^{r^{t+c}}) \ldots \ldots (1-Z^{r^{t+(b-1)c}})}{(1-Z^{r^{t-1}}) \ldots \ldots \ldots (1-Z^{r^{t-1+(b-1)c}})}$$

Man findet leicht dasz der Exponent von e gleich

$$(r^t - r^{t-1}) \frac{(1+r^{al^{h-1}} + \ldots + r^{(b-1)a^{ih-1}})}{l^h}$$

wird, also eine gerade ganze Zahl ist. Hieraus folgt:

$$\log \frac{A_t}{A_{t-1}} = \log \frac{(1-Z^{r^t})(1-Z^{r^{t+c}}) \ldots \ldots (1-Z^{r^{t+(b-1)c}})}{(1-Z^{r^{t-1}}) \ldots \ldots \ldots (1-Z^{r^{t-1+(b-1)c}})} = \log \varepsilon_t$$

Die Factoren 2, welche sich vor dem Producte aus (7) vorfinden, bringen wir in die Elemente der Determinante. Diese werden dann $log\, \varepsilon_t$ anstatt $log\, \varepsilon_t^2$ wie sie bis jetzt waren. Nachdem man noch Zeilen verwechselt hat, findet man die ganze, im Theorem dargestelte Form.

2. *Die Klassenanzahl für imaginäre Teilkörper (b ungerade).*

Der Hülfssatz I ergibt in diesem Falle

$$F\left(e^{\frac{2k\pi i}{l^h}}\right) = (-1)^u F\left(e^{-\frac{2k\pi i}{l^h}}\right)$$

Wir zerlegen das Product, welches von (4) genommen werden musz, in zwei Factoren: 1. das Product über alle geraden Werte von u; 2. das Product über alle ungeraden Werte von u.

Das erstgenannte Product kann man auf dieselbe Art umformen wie im Falle dasz b gerade ist, wegen

$$F\left(e^{\frac{2k\pi i}{l^h}}\right) = F\left(e^{-\frac{2k\pi i}{l^h}}\right)$$

Für dieses Product findet man daher

$$\leq \prod_{u \text{ gerade}}^{al^{h-1}-1} - \frac{1}{l^h} \sum_{k=1}^{l^h-1} F\left(e^{\frac{2k\pi i}{l^h}}\right) log\, A_k$$

Fassen wir nun weiter das zweite Product ins Auge. Es ist, ebenso wie früher schon gefunden ist:

$$\sum_{k=1}^{l^h-1} F\left(e^{\frac{2k\pi i}{l^h}}\right) = 0$$

und weiter:

$$\sum_{k=1}^{l^h-1} F\left(e^{\frac{k\pi i}{l^h}}\right) log\, A_k = \sum_{k=1}^{l^h-1} F\left(e^{\frac{2(l^h-k)\pi i}{l^h}}\right) log\, A_{l^h-k} =$$

$$= \sum_{k=1}^{l^h-1} F\left(e^{-\frac{2k\pi i}{l^h}}\right) log\, A_k = -\sum_{k=1}^{l^h-1} F\left(e^{\frac{2k\pi i}{l^h}}\right) log\, A_k$$

Hieraus ergibt sich dasz diese Summe verschwindet. Wir finden daher für das Product:

$$\leq \prod_{u \text{ ungerade}}^{al^{h-1}-1} + \frac{\pi i}{l^h} \sum_{k=1}^{l-1} k\, F\left(e^{\frac{2k\pi i}{l^h}}\right)$$

Um die Zahl $\frac{1}{\varkappa}$ zu bestimmen bemerken wir dasz der Körper

imaginär ist. Und da es ein Galois'scher Körper ist, sind alle conjugirte Körper imaginär, d.h. $r_1 = 0$, $r_2 = \frac{1}{2} a l^{h-1}$. Wiederum ist $w = 2$ (als $b \neq 1$ ist). Wir finden, nachdem die Werte substituirt sind:

$$H = \frac{|\sqrt{D}| \cdot \prod_{u \text{ ungerade}} \frac{\pi i}{l^{2h}} \sum_k k F\left(e^{\frac{2\pi k i}{l^h}}\right)}{2^{\frac{1}{2} a l^{h-1} - 1} \pi^{\frac{1}{2} a l^{h-1}}} \cdot \frac{\prod_{u \text{ gerade}} -\frac{1}{l^h} \sum_k F\left(e^{\frac{2\pi k i}{l^h}}\right) \log A_k}{R} \quad (8)$$

Die Factoren der beiden Producten werden auf dieselbe Art umgeformt wie für b gerade. Im ersten Product erscheinen dann die Factoren

$$S = \sum_k \left[\frac{k/l^{h'}}{l^h}\right]^{-bu} k$$

wo u teilbar ist durch $l^{h'}$ und nicht durch $l^{h'+1}$ und die Summe erstreckt sich über alle Werte von k die teilbar sind durch $l^{h'}$ und nicht durch $l^{h'+1}$. Die Summe S formen wir auf folgende Weise um:

$$S = l^{h1} \sum_k \left[\frac{k/l^{h'}}{l^h}\right]^{-bu} \frac{k}{l^{h'}} = \sum_k \left[\frac{k/l^{h'}}{l^h}\right]^{-bu} \frac{k}{l^{h'}} + \sum_k \left[\frac{k/l^{h'}}{l^h}\right]^{-bu} \left(\frac{k}{l^{h1}} + l^{h-h'}\right) + \ldots +$$

$$+ \sum_k \left[\frac{k/l^{h'} + (b^{h'}-1) l^{h-h'}}{l^h}\right]^{-bu} \left(\frac{k}{l^{h'}} + (l^{h'}-1)\, l^{h-h'}\right) -$$

$$- \{l^{h-h'} + 2\, l^{h-h'} + \ldots + (l^{h'}-1)\, l^{h-h'}\} \sum_k \left[\frac{k/l^{h'}}{l^h}\right]^{-bu}$$

Uebrigens ist

$$\sum_k \left[\frac{k/l^{h'}}{l^h}\right]^{-bu} = 0$$

da es die Summe aller $(l-1)l^{h-h'-1}$-ten Einheitswurzeln ist, multicirt mit einer ganzen Zahl.

Weiter sind nicht zwei der Zahlen $k/l^{h'} - i\, l^{h-1} \pmod{l^h}$ mit einander congruent. Keine dieser Zahlen ist teilbar durch l und ihre Anzahl ist gleich $\varphi(l^h)$. Man findet also

$$\sum_k k \left[\frac{k/l^{h'}}{l^h}\right]^{-bu} = \sum_k k \left[\frac{k}{l^h}\right]^{-bu}$$

wo in der letzten Summe die Zahl k sich erstreckt über alle Werte $< l^h$ welche nicht durch l teilbar sind.

Nach Umformung aller Factoren findet man sodann, wenn man bemerkt das a gerade ist und dasz das erste Product $\frac{1}{2} a l^{h-1}$ Factoren enthält und das zweite $\frac{1}{2} a l^{h-1} - 1$:

$$H = \frac{(-1)^r i^s \cdot \prod_{u \text{ ungerade}} \sum_{k=1}^{l^h - 1} \left[\frac{k}{l^h}\right]^{bu} k}{2^{\frac{1}{2} a l^{h-1} - 1} l^{\frac{1}{2} h a l^{h-1}}} \cdot \frac{\prod_{u \text{ gerade}} \sum_{k=1}^{l^h - 1} \left[\frac{k}{l^h}\right]^{bu} \log A^k}{R}$$

Die im ersten Factor auftretenden Summen kan man auf folgende Weise umformen:

$$S = \sum_{k=1}^{l^h} \left[\frac{k}{l^h}\right]^{bu} k = \sum_{k} e^{\frac{2\pi\, bu\, ind\, ki}{l^{h-1}(l-1)}} k.$$

Es sei $ind\, k = t$ so ist $k \equiv r^t \equiv r_t \pmod{l^h}$ wo r_t den kleinsten positiven Rest bedeutet von $r^t \pmod{l^h}$. Dann wird

$$S = \sum_{k=1}^{l^{h-1}(l-1)} e^{\frac{2\pi\, u\, t\, i}{al^{h-1}}} r_t.$$

Wenn man diese Summe verteilt in andere die sich erstrecken von $k = 1$ bis $k = al^{h-1}$, so ergibt sich:

$$S = \sum_{t=1}^{al^{h-1}} e^{\frac{2\pi\, u\, t\, i}{al^{h-1}}} \sum_{j=0}^{b-1} r_{t+j\, al^{h-1}}$$

Es ist leicht ersichtlich dasz die Summe $\sum_{j}$ für jeden Wert von t teilbar ist durch l^h.

Die Summe des zweiten Productes lässt sich wie folgt umformen:

$$\sum_{k=1}^{l^h-1} \left[\frac{k}{l^h}\right]^{bu} log\, A_k = \sum e^{\frac{2\pi ui\, ind\, k}{al^{h-1}}} log\, A_k =$$

$$= \sum_{t=1}^{\varphi(l^h)} e^{\frac{2\pi uti}{al^{h-1}}} log\, A_{r^t} = \sum_{t=1}^{al^{h-1}} e^{\frac{2\pi uti}{al^{h-1}}} log\, \mathrm{A}_t =$$

$$= \sum_{t=1}^{\frac{1}{2}al^{h-1}} + \sum_{t=\frac{1}{2}al^{h-1}+1}^{al^{h-1}} = \sum_{t=1}^{\frac{1}{2}al^{h-1}} e^{\frac{2\pi uti}{al^{h-1}}} log\, \mathrm{A}_t + \sum_{t=1}^{\frac{1}{2}al^{h-1}} e^{\frac{2\pi uti}{al^{h-1}}} log\, \mathrm{A}_{t+\frac{1}{2}al^{h-1}}$$

(weil u gerade ist)

$$= \sum_{t=1}^{\frac{1}{2}al^{h-1}} e^{\frac{2\pi uti}{al^{h-1}}} log\, \mathrm{A}_t\, \mathrm{A}_{t+\frac{1}{2}al^{h-1}}$$

Auf gleiche Weise, wie im Falle dasz b gerade ist, kann man diese Summe in eine Determinante verwandlen. Wenn man dann die Factoren 2 des Nenners aus (8) in die Elemente der Determinante hinein bringt, so findet man die im Theorem aufgegebene Determinante. Man musz nun noch die Potenzen von —1 beachten, und von i, die enstehen bei der Benutzung des Hülfssatzes IV.

Wenn wir schliesslich noch ein System reeller Grundeinheiten nehmen, so ist

$$log\ \eta_i^{(k)} = \tfrac{1}{2} l_k \cdot (\eta_i)\ ^1)$$

Wir müszen also in R substituiren $l_k(\eta_i) = 2 log\ \eta_i^{(k)}$ wonach die Potenz von 2 im Nenner verschwindet.

---

[1]) H. Seite 215.

Mathematics. — "*Ueber die Teilkörper des Kreiskörpers* $k\left(e^{\frac{2\pi i}{l^h}}\right)$." (Dritter Teil) [1]). By Dr. N. G. W. H. BEEGER. (Communicated by Prof. W. KAPTEYN).

(Communicated in the meeting of November 1918).

Der Ausdruck für die Klassenanzahl der primären Teilkörper hat die Form eines Productes zweier Brüche, wenn b eine ungerade Zahl ist. Ebenso wie es der Fall ist mit der Klassenanzahl des Kreiskörpers der l^h-ten Einheitswurzeln selbst, sind auch hier die zwei Brüche ganze Zahlen. Ich will dies hier beweisen, und nenne dabei die Brüche den ersten, resp. zweiten, Factor der Klassenanzahl.

Zuerst beweise ich einen Hülfssatz.

Hülfssatz 1.

Jedes System von Grundeinheiten eines primären Teilkörpers, in Bezug auf welches der Kreiskörper selbst den Relativgrad $2b$ hat, ist auch ein System von Grundeinheiten des primären Teilkörpers, in Bezug auf welches der Kreiskörper selbst den Relativgrad b hat, wenn b eine ungerade Zahl ist.

Beweis:

Wir nennen den ersten Teilkörper k und den zweiten K. Es ist dann k ein Teilkörper von K. Der Grad von K ist c; der von k ist dann $\frac{1}{2}c$ und $bc = \varphi(l^{h-1})$. Wir zeigen erst dasz k in K zu der Substitution

$$s^{1/2bc}, \text{ wo } s = (Z : Z^r), \text{ gehört.}$$

(r eine primitive Wurzel von l^h).

Die, den Körper K erzeugende, Zahl ist:

$$\eta_K = Z^{r^c} + Z^{r^{2c}} + \ldots + Z^{r^{bc}}$$

Also ist

$$s^{1/2bc}\,\eta_K = Z^{r^{c+1/2bc}} + Z^{r^{2c+1/2bc}} + \ldots + Z^{r^{bc+1/2bc}} \quad . \quad . \quad . \quad (1)$$

Nach einer kleinen Rechnung ergibt sich:

$$s^{1/2bc}\,\eta_K = Z^{r^{1/2c}} + Z^{r^{3/2c}} + \ldots + Z^{r^{(2b-1)1/2c}}$$

Die, den Körper k erzeugende, Zahl ist:

$$\eta_k = Z^{r^{1/2c}} + Z^{r^{2\cdot 1/2c}} + \ldots + Z^{r^{(2b-1)1/2c}}$$

[1]) Fortsetzung von „Proceedings" XXI Seite 454 und Seite 758.

Man sieht also dasz

$$\eta_k = \eta_K + s^{1/2bc}\,\eta_K$$

woraus sich ergibt dasz die Zahl η_k durch die Substitution $s^{1/2bc}$ nicht geändert wird. Es ist daher k der Teilkörper von K, der zu dieser Substitution gehört. Man folgert nun aus (1) dasz $s^{1/2bc}\eta_K$ conjugirt complex ist mit η_K da

$$r^{1/2bc} \equiv -1 \pmod{l^h}.$$

Es sei jetzt E eine Einheit von K, dann ist also $s^{1/2bc}E$ conjugirt complex mit E, und daher

$$\frac{E}{s^{1/2bc}E}\,' = 1$$

$\dfrac{E}{s^{1/2bc}E}$ ist also eine Einheit, die selbst, und deren Conjugirte sämtlich den absoluten Betrag 1 besitzen. Man weisz dasz eine solche Einheit immer eine Einheitswurzel ist [1]). Im Körper K bestehen aber nur die Einheitswurzeln ± 1. Man schlieszt hieraus

$$\frac{E}{s^{1/2bc}E} = \pm 1 \quad \ldots\ldots\ldots\ldots \quad (2)$$

Wenn nun $E = -s^{1/2bc}E$ wäre, so würde sich ergeben dasz der reelle Teil von E gleich Null wäre, und also E das Product einer reellen Einheit mit der Zahl i. E^2 würde also reell sein.

Setzen wir $E^2 = \varepsilon$, $E = \sqrt{\varepsilon}$. Die Relativdifferente der Zahl E in Bezug auf den Körper k ist daher $2\sqrt{\varepsilon}$, da die relativ conjugirte Zahl von $\sqrt{\varepsilon}$ gleich $-\sqrt{\varepsilon}$ ist. Die Relativdifferente der Zahl E ist also nicht teilbar durch l. Die Relativdifferente des Körpers K in Bezug auf k kan daher auch nicht teilbar sein durch l.

Der Körper K ist relativ cyclisch in Bezug auf k. Ich benutze darum einen Satz [2]) der relativ cyclischen Körper des Relativgrades 2: Es sei $\mathfrak{l}$ ein in l aufgehendes Primideal des Körpers k, dann ist $\mathfrak{l}$ nicht teilbar auf die Relativdifferente von K. Man schlieszt hieraus dasz die Primzahl l in K nicht teilbar ist durch das Quadrat eines Primideals. Nun gilt aber in k die Zerlegung

$$l = \mathfrak{l}^{1/2c}$$

und in K: $\quad \mathfrak{l} = \mathfrak{l}_1^{\,c}$ wo $\mathfrak{l}_1 = (1-Z)^b\,\mathfrak{G}$

wie sich aus Satz 4 ergibt. Es ist weiter in K:

$$\mathfrak{l} = \mathfrak{l}_1^{\,x}$$

Hieraus folgt aber $x = 2$, und l ist in K wohl durch das Quadrat eines Primideals teilbar.

[1]) H. Satz 48. [2]) H. Satz 93.

Man schliesst hieraus dasz die Gleichheit

$$E = -\varepsilon^{1/2bc} E$$

nicht bestehen kann. Dann musz aber

$$E = \varepsilon^{1/2bc} E.$$

E ist daher eine Einheit des Körpers k.

Satz 1.

Der zweite Factor der Klassenanzahl von K stellt die Klassenanzahl des Körpers k dar, und ist also eine ganze Zahl.

Beweis: Aus Satz 6 (S. 461) ergibt sich dasz die Δ der beiden Körper einander gleich sind.[1]) Aus dem Hülfssatz folgt dasz auch $R = R'$.

Satz 2.

Der erste Factor der Klassenanzahl von K ist eine ganze Zahl.

Beweis: Es ist leicht ersichtlich dasz

$$r_t + r_{t+c} + \ldots + r_{t+(b-1)c}$$

teilbar ist durch l^h. Man braucht daher nur noch zu zeigen dasz

$$\prod_u \sum_{t=1}^{c} e^{\frac{2\pi u t i}{c}} (r_t + r_{t+c} + \ldots + r_{t+(b-1)c})$$

teilbar ist durch $2^{1/2c-1}$. Die Summe teilen wir in zwei Partialsummen:

$$\sum_{t=1}^{1/2c} \quad \text{en} \quad \sum_{t=1/2c+1}^{c}$$

Für die letzte kann man schreiben:

$$\sum_{t=1}^{1/2c} e^{\frac{2\pi u(t+1/2c)i}{c}} (r_{t+1/2c} + r_{t+3/2c} + \ldots + r_{t+(b-1/2)c})$$

Es ist aber:

$$r_{t+n} \equiv \rho^{t+n} \equiv -\rho^{t+n\pm 1/2bc} \equiv -r_{t+n\pm 1/2bc} \pmod{l^h}$$

und daher:

$$r_{t+n} = l^h - r_{t+n\pm 1/2bc}.$$

Nach einer kleinen Rechnung findet man dasz die in der letzten Summe auftretende Form zwischen ( ) gleich

$$b \cdot l^h - (r_t + r_{t+c} \ldots r_{t+(b-1)c})$$

---

[1]) Im Satz 6 musz, im Falle b ungerade, vor den ersten Factor der Klassenanzahl, $(-1)^{1/2a}$ gesetzt werden. Vor der Determinante soll nicht $(-1)^{1/8(a-2)(a-4)}$ stehen, sondern $(-1)^{1/8(c-4)(c-6)}$.

ist. Da u ungerade ist, ergibt sich nun für die im Producte auftretende Summe:

$$2\sum_{t=1}^{1/2c} e^{\frac{2\pi uti}{c}} (r_t + r_{t+c} + \ldots + r_{t+(b-1)c}) + 2b \,.\, l^h \sum^{1/2c} e^{\frac{2\pi uti}{c}}$$

Es ist also, nachdem man die letztere dieser beiden Summen berechnet hat:

$$(1 - e^{\frac{-2\pi ui}{c}}) \sum_{t=1}^{c} e^{\frac{2\pi uti}{c}} (r_t + r_{t+c} + \ldots + r_{t+(b-1)c}) =$$

$$= 2 \left\{ b \,.\, l^h + \sum_{t=1}^{1/2c} e^{\frac{2\pi uti}{c}} (r_t + \ldots + r_{t+(b-1)c}) \right\}$$

Met Rücksicht auf die Beziehung

$$\prod_u (1 - e^{\frac{-2\pi ui}{c}}) = 2$$

findet man dasz das Product, welches im Zähler des ersten Factors der Klassenanzahl auftritt, teilbar ist durch $2^{1/2c-1}$.

Man kann den Beweis dieses Satzes auch erbringen auf die nämliche Art wie KUMMER und KRONECKER bewiesen haben dasz der erste Factor der Klassenanzahl eines jeden Kreiskörpers (der m-ten Einheitswurzeln) eine ganze Zahl ist, oder ein Bruch mit Nenner gleich 2. [1])

Verzeichnis einiger Werte des ersten Factors der Klassenanzahl.

l	h	b	erster Factor.
7	1	3	1
7	2	3	1
11	1	5	1
13	1	3	1
19	1	3	1
23	1	11	3
29	1	7	1
31	1	15	3
37	1	9	1
37	1	3	1
47	1	23	5

[1]) Monatsber. Berlin 1863 und Crelle's Journ. Band 40.

Satz 3.

Der erste Factor der Klassenanzahl des Teilkörpers, für welchen die Zahl b ungerade ist, ist ein Teiler der Klassenanzahl des Kreiskörpers (der l^h-ten Einheitswurzeln) selbst.

Beweis: Der erste Factor der Klassenanzahl des Kreiskörpers selbst, ist:[1])

$$\frac{\prod_{u}\sum_{t=1}^{l^{h-1}(l-1)} e^{\frac{2\pi uti}{l^{h-1}(l-1)}r_t}}{2^{1/2\varphi-1}l^{h(1/2\varphi-1)}},$$

wo, zur Abkürzung φ den Wert der Function $\varphi(l^h)$ vorstellt, und das Product sich erstreckt über alle Zahlen 1, 3, ... $\varphi-1$. Man weisz dasz der Bruch einer ganzen Zahl gleich ist. Die im Zähler auftretende Summe bringen wir in die folgende Form

$$\sum_{t=1}^{c}\left\{e^{\frac{2\pi uti}{\varphi}}r_t + e^{\frac{2\pi u(t+c)i}{\varphi}}r_{t+c} + \ldots + e^{\frac{2\pi u(t+(b-1)c)i}{\varphi}}r_{t+(b-1)c}\right\}.$$

Für die Werte

$$u = bu' \quad u' = 1, 3, \ldots . c-1$$

hat diese Summe denselben Wert wie die Summe welche auftritt im Zähler des ersten Factors der Klassenanzahl des Teilkörpers des Relativgrades b. Der letztgenannte Zähler ist daher ein Teiler des Zählers des Kreiskörpers selbst. Der Quotient der beiden ersten Factoren der Klassenanzahlen ist nun

$$\frac{1}{2^{1/2\varphi-1/2c}\,l^{h(1/2\varphi-1/2c-1)}}\prod_{u}\sum_{t=1}^{\varphi} e^{\frac{2\pi uti}{\varphi}}r_t \quad . \quad . \quad . \quad . \quad . \quad (3)$$

wo u alle ungeraden Zahlen 1, 3,.... $\varphi-1$ durchläuft die nicht durch b teilbar sind. Es ist weiter:

$$(re^{\frac{2\pi ui}{\varphi}}-1)\sum_{t=1}^{\varphi} e^{\frac{2\pi uti}{\varphi}}r = \sum_{t=1}^{\varphi} e^{\frac{2\pi u(t+1)i}{\varphi}}(rr_t - r_{t+1})$$

und $rr_t - r_{t+1} \equiv 0 \pmod{l^h}$.

Auch ist:

$$\prod_{u}(re^{\frac{2\pi ui}{\varphi}}-1) = \frac{\prod_{u}(re^{\frac{2\pi ui}{\varphi}}-1}{\prod_{u'}(re^{\frac{2\pi u'i}{\varphi}}-1)} = \frac{r^{\frac{1}{2}\varphi}+1}{r^{\frac{1}{2}c}+1}$$

[1]) H. Satz 141.

wo, im ersten Product, u alle Werte $1, 3, \ldots . \varphi - 1$ durchläuft die nicht durch b teilbar sind; im zweiten Product aber, durchläuft u alle Zahlen $1, 3, \ldots . \varphi - 1$. Der Zähler des letzten Bruches ist teilbar durch l^h; der Nenner nicht. Man kann die primitive Wurzel r so wählen dasz der Zähler nicht teilbar ist durch l^{h-1}. Hieraus ergibt sich dasz der Zähler aus (3) durch die Potenz von l teilbar ist, die sich im Nenner von (3) vorfindet.

Auf dieselbe Art wie beim Beweise des vorigen Satzes kann man beweisen dasz der Zähler von (3) teilbar is durch die Potenz von 2 die sich im Nenner vorfindet.

Auf ganz gleiche Art kann man den Satz beweisen:

Wenn ein Teilkörper k Teilkörper ist eines Teilkörpers K, so ist der erste Factor der Klassenanzahl von k ein Teiler des ersten Factors von K. (Der Relativgrad b ist für beide Körper also ungerade anzunehmen).